PHY306 Introduction to Cosmology
Practice Problems

NOTE:

The Friedman equation: $$\dot{a}(t)^2 = \frac{8\pi G}{3c^2}\left(\frac{\mathcal{E}_{\mathrm{r0}}}{a(t)^2} + \frac{\mathcal{E}_{\mathrm{m0}}}{a(t)}\right) - \frac{kc^2}{R_0^2} + \frac{\Lambda}{3}a(t)^2$$

The fluid equation: $$\dot{\mathcal{E}} + 3\frac{\dot{a}}{a}(\mathcal{E} + P) = 0$$

The acceleration equation: $$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}(\mathcal{E} + 3P)$$

The Robertson-Walker metric: $$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a(t)^2\left(\mathrm{d}r^2 + x_k^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right)$$

$$\text{where } x_k(r) = \begin{cases} R\sin(r/R) & k = +1 \\ r & k = 0 \\ R\sinh(r/R) & k = -1 \end{cases}$$

Useful constants:

$c = 2.998 \times 10^8$ m s^{-1}. $G = 6.674 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.

1 pc $= 3.086 \times 10^{16}$ m. $M_\odot = 1.989 \times 10^{30}$ kg.

1 yr $= 3.156 \times 10^7$ s. 1 GeV $= 1.602 \times 10^{-10}$ J.

1 The Robertson-Walker Metric

1. The RW metric can be written in terms of comoving polar coordinates (x, θ, ϕ) as

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left(\frac{\mathrm{d}x^2}{1 - kx^2/R^2} + x^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right).$$

   (a) What is the significance of (i) k and (ii) R?

   (b) Show, by integrating over a radial displacement $\mathrm{d}x$, that the comoving proper distance r is given by

      (i) $r = R\sinh^{-1}(x/R)$, for a universe with open geometry;

      (ii) $r = x$, for a flat universe;

      (iii) $r = R\sin^{-1}(x/R)$, for a universe with closed geometry.

   (c) With reference to this, explain why the curvature of the universe is not significant for $r \ll R$.

   (d) Using the above expressions, justify the use of the terms 'open' and 'closed' geometry.

2. By considering a photon moving in a radial direction, show that the Robertson-Walker metric implies that the comoving proper distance r is given by

$$r = c\int_{t_e}^{t_o} \frac{\mathrm{d}t}{a(t)},$$

   where t_e is the time at which the photon is emitted, and t_o the time at which it is observed.

   Hence show that the observed and emitted wavelengths, λ_o and λ_e, are related by

$$\frac{\lambda_o}{\lambda_e} = \frac{a(t_o)}{a(t_e)}.$$

3. (a) The **luminosity distance** d_L of an object is defined by

$$f = \frac{L}{4\pi d_\mathrm{L}^2},$$

where L is the luminosity of the object and f is the observed flux. (This is equivalent to defining d_L, in parsecs, by $m - M = 5\log(d_\mathrm{L}) - 5$, where m is the apparent magnitude of the object and M is its absolute magnitude.)

   (i) Use the Robertson-Walker metric to show that a sphere of comoving proper radius r has surface area $4\pi x_k^2$, where x_k is defined on the front cover.

   (ii) Use the result of question 2 to show that the observed luminosity, $4\pi x_k^2 f$, is less than the emitted luminosity L by a factor of $(1+z)^2$.

   (iii) Hence deduce that $d_\mathrm{L} = x_k(1 + z)$.

(b) The **angular diameter distance** d_A of an object is defined by

$$\theta = \frac{\ell}{d_\mathrm{A}},$$

where ℓ is the linear size of the object and θ is its observed angular size.

   (i) Use the Robertson-Walker metric to show that a small perpendicular length ℓ at comoving proper distance r can be given by $\ell^2 = a^2 x_k^2 \mathrm{d}\theta^2$. [Hint: as ℓ is small, no integration is necessary. You are always free to define the polar coordinate system such that ℓ lies along the ϕ axis.]

   (ii) Hence deduce that $d_\mathrm{A} = x_k/(1 + z)$.

4. (a) Systematic errors on distances are typically of order 5% or more. At what redshift is it necessary to start paying attention to whether your distance indicator measures luminosity distance (standard candle) or angular diameter distance (standard ruler)?

(b) For the local universe ($z \ll 1$), the Hubble law can be written $cz = H_0 d$, where d is the distance.

   (i) Galaxies have typical peculiar velocities of order 1000 km s^{-1}. If we assume that H_0 is in the range 50–100 km s^{-1} Mpc^{-1}, at what distance do we need to work in order to ensure that galaxy peculiar velocities introduce errors of less than 10%?

   (ii) On this distance scale, do we need to worry about whether the d in the above equation is a luminosity distance or an angular diameter distance? Justify your answer.

2 The Friedman Equation

Note that in this and all subsequent sections, the subscript 0 refers to the value of the quantity at the present time.

5. The Friedman equation can be written as

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\mathcal{E}}{3c^2} - \frac{kc^2}{R_0^2 a^2},$$

   where $\mathcal{E}$ is the energy density and k and R_0 have their usual meanings.

   Use this and the fluid equation to derive the acceleration equation.

6. (a) Use the fluid equation to show that

$$\mathcal{E} = \mathcal{E}_0 a^{-3(1+w)}$$

   for a substance whose equation of state is $P = w\mathcal{E}$.

   (b) Hence derive the a-dependences of (i) matter, (ii) radiation, and (iii) a cosmological constant, stating any assumptions that you make.

   (c) Using the results you have obtained, derive the form of the Friedman equation given on the front cover from the form given in question 1. Include in your answer an expression for the relation between the vacuum energy density $\mathcal{E}_\Lambda$ and the cosmological constant Λ.

7. Give an expression for the **critical density**. For $H_0 = 70$ km s^{-1} Mpc^{-1}, calculate the critical density in (i) J m^{-3}; (ii) kg m^{-3}; (iii) $M_\odot$ Mpc^{-3}; (iv) GeV m^{-3}.

8. Find the value of the radius of curvature, R_0, if $\Omega_{k0} = -0.09$ (where $\Omega_{k0} = 1 - (\Omega_{r0} + \Omega_{m0} + \Omega_{\Lambda 0})$) and $H_0 = 50$ km s^{-1} Mpc^{-1}. Hence determine the maximum proper distance r for which the difference between r and x_k is less than 5% (for these values of Ω_k and H_0).

   (Note that Ω_k and H_0 are strongly correlated in fits to the CMB. Figure 21 of the Planck cosmological results paper (arXiv:1303.5076) shows that a value of 50 km s^{-1} Mpc^{-1} for H_0 is about right for this value of Ω_k.)

9. Define the **density parameter** Ω. Hence derive the following forms of the Friedman equation:

   (i) in terms of Ω_i (where i can be matter, radiation or Λ),

$$H^2\left(1-\Omega_{\rm r}-\Omega_{\rm m}-\Omega_\Lambda\right)=-\frac{kc^2}{a^2R_0^2};$$

   (ii) in terms of Ω_{i0},

$$\dot{a}^2=H_0^2\left(\frac{\Omega_{\rm r0}}{a^2}+\frac{\Omega_{\rm m0}}{a}+\Omega_{k0}+\Omega_{\Lambda 0}a^2\right), \qquad (1)$$

   including in your answer the definition of Ω_{k0}.

10. Show that

$$1-\Omega(t)=\frac{H_0^2(1-\Omega_0)}{H^2a^2}, \qquad (2)$$

    where $\Omega=\Omega_{\rm r}+\Omega_{\rm m}+\Omega_\Lambda$.

11. (a) Show that, with the above definition of Ω, the sign of k is entirely determined by the sign of $1-\Omega$ and vice versa.

    (b) Using equation 1, show that a universe in which $k=+1$ and $\Lambda=0$ will reach a maximum value of a and then recollapse.

    (c) By differentiating equation 1 in the case where the universe is flat and the radiation density is negligible, find the condition under which the expansion of the universe, $\dot{a}$, will be accelerating.

12. (a) Show that, (almost) regardless of the current values of $\Omega_{\rm r,m,\Lambda}$, the universe described by equation 1 will be radiation dominated at early times and dominated by Λ at late times. What is the one situation in which this is not true?

    (b) Determine (in terms of $\Omega_{\rm r0}$ and $\Omega_{\rm m0}$) the redshift z at which $\Omega_{\rm r}=\Omega_{\rm m}$ (the **epoch of matter-radiation equality**).

    (c) Determine (in terms of $\Omega_{\rm m0}$ and $\Omega_{\Lambda 0}$) the redshift z at which $\Omega_{\rm m}=\Omega_\Lambda$.

13. (a) The total radiation energy density of blackbody radiation at temperature T is given by

$$\mathcal{E}_{\rm BB} = \frac{4\sigma}{c} T^4,$$

where σ is Stefan's constant (5.67×10^{-8} W m^{-2} K^{-4}). If $\Omega_{\rm m0} = 0.28$ and $H_0 = 74$ km s^{-1} Mpc^{-1}, calculate the redshift at which $\Omega_{\rm m}(z) = \Omega_{\rm r}(z)$, given that the present temperature of the cosmic microwave background is 2.7 K.

The answer you get is in fact wrong: WMAP finds that the redshift of matter-radiation equality is $z_{\rm eq} = 3200$. What did we neglect in our calculation above?

(b) Assuming $k = 0$ and the other parameters as given in the previous part, calculate the redshift at which the expansion of our universe began to accelerate. You will need the result from question 11.

2.1 Single-component solutions of the Friedman equation

14. Show that, in a flat universe with zero matter density and no cosmological constant, $a(t) = (t/t_0)^{1/2}$, stating your boundary condition and any other assumptions made.

Hence derive

(i) the age of this universe in terms of the Hubble constant H_0;

(ii) the comoving proper distance r of an object at redshift z;

(iii) the comoving horizon distance, i.e. the distance $r_{\rm hor}$ at which $z = \infty$.

15. Show that, in a flat universe with zero radiation density and no cosmological constant, $a(t) = (t/t_0)^{2/3}$, stating your boundary condition and any other assumptions made.

Hence derive

(i) the age of this universe in terms of the Hubble constant H_0;

(ii) the comoving proper distance r of an object at redshift z;

(iii) the comoving horizon distance, i.e. the distance $r_{\rm hor}$ at which $z = \infty$.

16. Show that, in a flat universe with zero radiation and matter density but a positive cosmological constant Λ, $a(t) = \exp(H(t - t_0))$, stating your boundary condition and any other assumptions made, and including an expression for H in terms of Λ.

    Hence derive

    (i) the comoving proper distance r of an object at redshift z;

    (ii) the comoving horizon distance, i.e. the distance r_{hor} at which $z = \infty$.

    Comment on the age of this universe, and relate this to the result you obtain for the horizon distance.

17. Consider a universe with zero energy density ($\Omega_\text{r} = \Omega_\text{m} = \Omega_\Lambda = 0$).

    (a) Show that this universe cannot have $k = +1$.

    (b) Solve the Friedman equation for the case $k = 0$ and explain its physical significance.

    (c) Solve the Friedman equation for the case $k = -1$. Obtain expressions for the age of this universe, the comoving proper distance of an object at redshift z, and the comoving horizon distance.

18. The metal-poor star HE 1523–0901 has an age of 13.4 ± 2.2 Gyr as determined from uranium-based radiochemical dating (A Frebel et al., *ApJ* **660** (2007) L117). Assuming a matter-only model, calculate the value of H_0 that this would imply, if the first stars form 100 Myr after the Big Bang. Comment on your result.

19. If the universe is very nearly flat, $\Omega_0 \simeq 1$, obtain an expression for $H(t)$ in terms of H_0 and a for

    (i) a matter-dominated universe, $\Omega_0 = \Omega_{\text{m}0}$;

    (ii) a radiation-dominated universe, $\Omega_0 = \Omega_{\text{m}0}$;

    (iii) a Λ-dominated universe, $\Omega_0 = \Omega_{\Lambda 0}$.

    Hence use equation 2 to find how $1 - \Omega(t)$ depends on $a(t)$ for each of these scenarios, and comment on your result.

2.2 The Open Universe

A "matter plus curvature" open universe $(\Omega_{r0} = \Omega_{\Lambda 0} = 0,\ \Omega_{m0} < 1)$ *was a popular alternative to the "Standard Cold Dark Matter" (SCDM) model* $(\Omega_{m0} = 1)$ *in the 1990s.*

20. The Friedman equation for this model can be written

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_{m0}}{a} + (1 - \Omega_{m0}) \right). \tag{3}$$

   (a) By integrating the Friedman equation using the substitution $\sinh \frac{1}{2}\theta = \sqrt{Qa}$, where $Q = (1 - \Omega_{m0})/\Omega_{m0}$, show that the solution can be written in the parametric form

$$\begin{aligned} a(\theta) &= \frac{\Omega_{m0}}{2(1 - \Omega_{m0})}(\cosh\theta - 1) \\ t(\theta) &= \frac{\Omega_{m0}}{2H_0(1 - \Omega_{m0})^{3/2}}(\sinh\theta - \theta) \end{aligned} \tag{4}$$

   where θ is a dummy variable.

   (b) Conversely, show by differentiation that the parametric equations above are indeed a solution of equation 3. [*Note: this is easier than the previous part. Try it even if you couldn't do the integration.*]

   (c) Show that the solution given by equations 4 tends to the matter-only expression $a(t) = (t/t_0)^{2/3}$, where $t_0 = \frac{2}{3}H_0^{-1}$, in the limit $\Omega_{m0} \to 1$.

   (d) Show that it tends to the empty open universe solution $a(t) = H_0 t$ in the limit $\Omega_{m0} \to 0$.

21. Using equations 4, find, for a model in which $H_0 = 74\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $\Omega_{m0} = 0.28$,

   (i) the age of the universe;

   (ii) the **look-back time** $t_0 - t_e$ for an object at redshift 3.0.

   Calculate the same quantities for an SCDM (matter-only) model with the same value of H_0, and compare.

22. Using equations 4, find the comoving proper distance r for this model in terms of c, H_0, $\Omega_{\mathrm{m}0}$ and the parameter θ. Hence calculate, for an open model in which $H_0 = 74\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ and $\Omega_{\mathrm{m}0} = 0.28$,

    (i) the comoving horizon distance;

    (ii) the comoving proper distance to an object at redshift 3.0.

    Again, compare your answers with the corresponding results for the SCDM model with the same H_0.

23. (a) As mentioned in question 18, the metal-poor star HE 1523–0901 has an age of 13.4 ± 2.2 Gyr. Again assuming that the first stars form 100 Myr after the Big Bang, calculate the value of H_0 implied for an open model with $\Omega_{\mathrm{m}0} = 0.28$. Compare your answer with the one you obtained in question 18, and with the value of $74.2 \pm 3.6\ \mathrm{km\,s^{-1}\,Mpc^{-1}}$ calculated by Riess et al. (*ApJS* **183** (2009) 109) using Type Ia supernovae.

    (b) Calculate the maximum value of $\Omega_{\mathrm{m}0}$ that would be consistent with the age of HE 1523–0901 *and* the value of H_0 found by Riess et al.
    [Hint: the best bet here is probably a spreadsheet. Find the range of $H_0 t_0$ that is acceptable, and then construct a spreadsheet that calculates $H_0 t_0$ as a function of $\Omega_{\mathrm{m}0}$, using θ as an intermediate parameter. Read off the appropriate value. (Calculating it analytically is possible in principle, but probably needs a computer algebra package! Trial and error will also work.)]

2.3 The Closed Universe

The "matter plus curvature" closed universe $(\Omega_{r0} = \Omega_{\Lambda 0} = 0,\ \Omega_{m0} > 1)$ *is similar to Einstein's original static model, but without the cosmological constant term opposing the expansion.*

24. The Friedman equation for this model can be written

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_{m0}}{a} - (\Omega_{m0} - 1) \right). \tag{5}$$

(a) By integrating the Friedman equation using the substitution $\sin \frac{1}{2}\theta = \sqrt{Qa}$, where $Q = (\Omega_{m0} - 1)/\Omega_{m0}$, show that the solution can be written in the parametric form

$$\begin{aligned} a(\theta) &= \frac{\Omega_{m0}}{2(\Omega_{m0} - 1)}(1 - \cos\theta) \\ t(\theta) &= \frac{\Omega_{m0}}{2H_0(\Omega_{m0} - 1)^{3/2}}(\theta - \sin\theta) \end{aligned} \tag{6}$$

where θ is a dummy variable.

(b) Conversely, show by differentiation that the parametric equations above are indeed a solution of equation 5. [*Note: this is easier than the previous part. Try it even if you couldn't do the integration.*]

(c) Show that the solution given by equations 6 tends to the matter-only expression $a(t) = (t/t_0)^{2/3}$, where $t_0 = \frac{2}{3}H_0^{-1}$, in the limit $\Omega_{m0} \to 1$.

(d) Show that the age of this universe is always less than the age of the matter-only universe, $t_0 = \frac{2}{3}H_0^{-1}$. (Note: this is done directly from equation 5—from equations 6 it's hard to do analytically, though straightforward numerically.)

(e) Using equations 6, find the comoving proper distance r for this model in terms of c, H_0, Ω_{m0} and the parameter θ.

25. A more general closed-geometry universe would include the cosmological constant and would have the form

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_{\mathrm{m}0}}{a} + \Omega_{\Lambda 0} a^2 + \Omega_{k0} \right), \tag{7}$$

where $\Omega_{k0} = 1 - \Omega_{\mathrm{m}0} - \Omega_{\Lambda 0} < 0$.

(a) Explain why a universe of this type does *not* necessarily recollapse, despite its closed geometry.

(b) Find, in terms of $\Omega_{\mathrm{m}0}$ and $\Omega_{\Lambda 0}$, the value of a at which the expansion of a universe with non-zero Λ begins to accelerate. Explain why your result does not depend on the curvature Ω_{k0}.

(c) Explain why, in spite of the above, a closed universe with non-zero Λ *may* recollapse, if Λ is small enough.

(d) Find an expression, in terms of $\Omega_{\mathrm{m}0}$ and $\Omega_{\Lambda 0}$, for the maximum value of a reached in equation 7 (it's effectively a cubic, and therefore difficult to solve analytically). Substitute in the value of a found in question 25b above, and hence derive an equation relating $\Omega_{\mathrm{m}0}$ and $\Omega_{\Lambda 0}$.

(e) (*Challenging!*)
If you expand the equation you derived above for the case in which $\Omega_{\Lambda 0} \ll \Omega_{\mathrm{m}0} - 1$, you get an equation which, given a value of $\Omega_{\mathrm{m}0}$, can be solved iteratively for $\Omega_{\Lambda 0}$. Do this and compare your results with, e.g., the Supernova Cosmology Project plot at
`http://supernova.lbl.gov/PDFs/SCP2003OmegaConfTrans.pdf`.

26. Equation 7 above describes the Einstein static universe if $\dot{a} = 0$ and $\ddot{a} = 0$.

(a) By going back to the front-cover form of the Friedman equation and assuming that $\mathcal{E}_{\mathrm{r}0}$ is negligible, show that the second condition requires

$$\Lambda = \frac{4\pi G \mathcal{E}_{\mathrm{m}0}}{c^2},$$

and deduce the radius of curvature of Einstein's universe.

(b) By considering the effect of a small change $\Delta\mathcal{E}$ in $\mathcal{E}_{\mathrm{m}}$ (leaving Λ and R_0 fixed at their previously tuned value), show that Einstein's model is unstable to small perturbations.

2.4 The Benchmark Universe

The current standard cosmological model, which I call the "benchmark" model to distinguish it from the Standard Model of particle physics, is a flat universe with $\Omega_{\rm r0} = 0$, $\Omega_{\rm m0} > 0$ *and* $\Omega_{\Lambda 0} = 1 - \Omega_{\rm m0}$. *The best current value of* $\Omega_{\rm m0}$ *(WMAP 9-year) is about 0.28—it depends slightly on which dataset you choose.*

27. The Friedman equation for this model can be written

$$\dot{a}^2 = H_0^2 \left(\frac{\Omega_{\rm m0}}{a} + (1 - \Omega_{\rm m0})\, a^2 \right). \tag{8}$$

(a) By integrating the Friedman equation using the substitution $\sinh\theta = \sqrt{Qa^3}$, where $Q = (1 - \Omega_{\rm m0}) / \Omega_{\rm m0}$, show that the solution can be written as

$$a = \left(\frac{\Omega_{\rm m0}}{1 - \Omega_{\rm m0}} \right)^{1/3} \sinh^{2/3}\left(\tfrac{3}{2} \sqrt{1 - \Omega_{\rm m0}} H_0 t \right). \tag{9}$$

(b) Conversely, show by differentiation that equation 9 is indeed a solution of equation 8.

(c) What is the age t_0 of this universe, in terms of H_0 and $\Omega_{\rm m0}$?

(d) Show that equation 9 tends to the matter-only solution $a(t) = (t/t_0)^{2/3}$, where $t_0 = \frac{2}{3} H_0^{-1}$, in the limit $\Omega_{\rm m0} \to 1$. What is the difference between this and the very similar solution for the limit $t \to 0$?

(e) Show that equation 9 tends to the expression

$$a(t) \simeq a_\Lambda e^{Ht}$$

in the limit $t \to \infty$, providing an expression for the constant a_Λ and justifying the use of H in the exponential. [Hints: consider what happens to $\sinh\theta$ for large θ, and $\dot{a}^2$ for large a.]

28. If we take $H_0 = 74$ km s^{-1} Mpc^{-1} and $\Omega_{\rm m0} = 0.28$, what is

(i) the age of the universe;

(ii) the look-back time, $t_0 - t_e$, to a quasar of redshift 3.0?

Compare the age you obtain with the age of the metal-poor star in question 18, and comment on the result.

29. (*Challenging!*)
The proper distance in this model is, to quote Barbara Ryden, "not a user-friendly integral", but it is not difficult to evaluate numerically. Construct a program or spreadsheet that does this, and calculate the horizon distance and the comoving proper distance to an object at redshift 3.0, using the values of H_0 and $\Omega_{\rm m0}$ in the previous question. Compare your answers to the matter-only universe with the same H_0, and to the open universe with the same H_0 and $\Omega_{\rm m0}$. Try modifying the input value of $\Omega_{\rm m0}$, and check that as it gets closer to 1 the result gets closer to the matter-only calculation (if it doesn't, there's a bug in your numerical integration!).

30. Show that the expansion $\dot{a}$ of the benchmark universe initially decelerates but subsequently accelerates. Find an expression for the redshift at which this switch happens. Calculate the numerical value of this redshift for $H_0 = 74$ km s^{-1} Mpc^{-1} and $\Omega_{\rm m0} = 0.28$. What is the look-back time to this redshift? Comment on your result.

31. For a matter-only universe, what value of H_0 would give the same age as you calculated for the benchmark universe in question 28i? Calculate the look-back times for this matter-only universe and the benchmark universe, for redshifts 1.0, 2.0, 3.0, 4.0 and 5.0, and comment on your results. If you did question 29, repeat this for the proper distances.

3 Observational Cosmology

32. Most distance indicators are "standard candle" methods using $m - M = 5\log d - 5$, where m is the measured apparent magnitude, M is the known absolute magnitude, and d is the (luminosity) distance in parsecs.

    (a) Considered as a method of determining H_0 from the Hubble law $cz = H_0 d$, this method is particularly vulnerable to systematic errors in M. Explain why.

    (b) Calculate the effect on H_0 of an error ΔM in M.

    (c) What is the effect on H_0 of a systematic error $\Delta\ell$ in the length of a "standard ruler"?

    [Note: in the above question, assume that we are working at $z \ll 1$, so we can neglect the difference between luminosity distance, angular diameter distance and proper distance.]

33. The intermediate-range distance indicators used for determining H_0 are not, in general, "primary" distance indicators, but require calibration.

    (a) Describe the properties of an ideal calibrating indicator, carefully justifying each requirement.

    (b) The most commonly used calibrator is the period-luminosity relation of classical Cepheids. With reference to your previous answer, discuss the extent to which classical Cepheids meet the specifications of an ideal calibrator.

    (c) In several recent papers, including Riess et al. (2009), the mega-maser galaxy NGC 4258 has been used as a calibrator in place of, or in addition to, classical Cepheids. Explain how this works as a distance indicator, and what its advantages and disadvantages are compared to classical Cepheids.

34. The intermediate-range distance indicators used by the Hubble Key Project (WL Freedman et al., *ApJ* **553** (2001) 47) were (i) the Tully-Fisher relation, (ii) the fundamental plane, (iii) surface brightness fluctuations and (iv) Type II supernovae. They also used (v) Type Ia supernovae, which are a long-range distance indicator (usable out to $z \sim 1$).

    (a) For each of these methods, explain how the method works, its approximate range, any potential systematic errors, and its principal advantages and disadvantages.

    (b) For each method, state whether it measures luminosity distance or angular diameter distance, explaining and justifying your choice. (Note: at least one method can in principle be either, depending on how it is used.)

    (c) The values of the Hubble constant found by Freedman et al. were as shown in the following table (the first error is random, the second systematic).

Indicator	H_0	Errors (%)
36 Type Ia supernovae	71	$\pm 2,\ \pm 6$
21 Tully-Fisher clusters	71	$\pm 3,\ \pm 7$
11 fundamental-plane clusters	82	$\pm 6,\ \pm 9$
SBF for 6 clusters	70	$\pm 5,\ \pm 6$
4 Type II supernovae	72	$\pm 9,\ \pm 7$

    It is reasonable to suspect that the systematic errors are not independent, but the random errors should be.

    (i) Calculate the weighted mean of the different values, with and without the one obvious outlier. (Note: as the systematic errors are not independent, don't use them for the weights. Weight by the random errors only.)

    (ii) Discuss whether there is good reason to suspect a problem with the outlying value—that is, is its difference from the others statistically significant?

    (iii) Comment on whether you might expect the outlying method to be less well calibrated or less reliable than the others.

    (d) Why did the Hubble Key Project team not use the classical Cepheids themselves in the determination of H_0?

35. Assume you are a cosmologist working about 1970. Like most mainstream cosmologists of the time, you are a supporter of the Hot Big Bang cosmological model.

    (a) Provide 3 pieces of evidence that support your position that the Hot Big Bang is the correct model. In each case, explain *why* the evidence supports this model in preference to others.

    (b) Provide 3 pieces of evidence that might cause you to worry that the Hot Big Bang does not explain *all* the available data.

36. Around 1950, your mother—also a cosmologist—supported the Steady State cosmological model.

    (a) At the time, it was generally accepted that $H_0 \sim 500$ km s^{-1} Mpc^{-1}, with an error estimated to be 10–20%. Explain why this value caused your mother to feel that the Big Bang model was probably wrong.

    (b) Outline the basic principles of the Steady State model.

    (c) Assuming, as one would in 1950, that $H_0 \sim 500$ km s^{-1} Mpc^{-1}, that $\Omega_{\rm r0} = \Omega_{\Lambda 0} = 0$, and that the geometry of the universe is flat, use the fluid equation to calculate the rate of matter creation expected in the Steady State model. Express your answer in atomic mass units per cubic metre per second, and comment on the result.

    (d) What are the *testable* predictions of the Steady State model?

    (e) Provide 3 pieces of evidence, emerging during the 1960s, that would have caused your mother to abandon her support of the Steady State model.

3.1 Inflation

37. What is meant by **inflation**? Briefly explain what problems with the classical Big Bang model inflation was intended to solve, and how it solves them.

38. According to the latest WMAP data, the epoch of radiation-matter equality occurred at $z_{\rm eq} \simeq 3280$, and the overall curvature of the universe is given by $\Omega_{k0} = 1 - \Omega_0 = -0.0027^{+0.0039}_{-0.0038}$.

    (a) Assume that the universe can be treated as matter-dominated from the present day back to $z_{\rm eq}$. Using the results of question 19, calculate the value of Ω_{k0} at $z_{\rm eq}$.

    (b) Assume that from $z_{\rm eq}$ back to the period of inflation ($t \sim 10^{-35}$ s) the universe can be treated as radiation-dominated. Calculate the value of Ω_{k0} at $t = 10^{-35}$ s. Assume that the time of matter-radiation equality is $t_{\rm eq} = 50\,000$ years.

    (c) Hence determine the minimum number of e-foldings of inflation needed to produce a universe at least as flat as ours, assuming that before inflation $|\Omega_{k0}| \sim 1$.

39. (a) Repeat the last question, but this time assume that the universe can be treated as matter-dominated throughout, i.e. use the matter-dominated a-dependence all the way from now back to $t = 10^{-35}$ s. Assume that the age of the universe now is 13.9 Gyr.

    (b) Do the same again, this time assuming that the universe can be treated as *radiation*-dominated throughout.

    (c) Which of the above calculations was the better approximation, i.e. which came closer to the more accurate calculation done in the previous question? Is this what you expected? Why is it the case?

40. Assume a matter-only universe with $H_0 = 50$ km s^{-1} Mpc^{-1} (to get roughly the right age).

    (a) (i) Calculate the horizon distance $d_{\rm hor}$ for this universe now.

    (ii) Calculate its horizon distance $d_{\rm CMB}$ at $z = 1100$, the redshift of the cosmic microwave background.

    (iii) Calculate the angle subtended by a length $\ell = d_{\rm CMB}$ at the proper distance corresponding to $z = 1100$. [Hint: refer to question 3.]

    (iv) Comment on your result.

    (b) (i) Calculate the horizon distance $d_{\rm inf}$ at $t = 10^{-35}$ s.

    (ii) What distance at $t = 10^{-35}$ s corresponds to the distance $d_{\rm hor}$ now? [Hint: what is the redshift corresponding to $t = 10^{-35}$ s?]

    (iii) Comparing the previous two answers, how many e-foldings of inflation would be needed to ensure that the present-day universe could originate from a single causally connected pre-inflation domain?

41. The equation of state of a scalar field φ is given by

$$\mathcal{E}_\varphi = \frac{1}{2\hbar c^3}\dot{\varphi}^2 + U(\varphi);$$

$$P_\varphi = \frac{1}{2\hbar c^3}\dot{\varphi}^2 - U(\varphi),$$

where the first term is the kinetic energy term and the second is the potential.

   (a) From the form of these equations, explain why, and under what assumption, a scalar field can be treated as a cosmological constant.

   (b) Briefly explain the form of the potential $U(\varphi)$ necessary to achieve inflation.

   (c) Explain why inflation does not generate a *completely* homogeneous and isotropic universe, but instead produces a universe with small variations in density. What property are these density variations expected to have?

3.2 The Benchmark Universe

42. The current benchmark model has the following properties:

- flat geometry ($k = 0$);
- negligible radiation energy density;
- matter density $\Omega_{\mathrm{m}0} \simeq 0.28$, of which only $\sim$0.046 is baryonic matter, the rest being cold dark matter;
- remaining energy density accounted for by a component whose properties are consistent with a cosmological constant/vacuum energy density, $\Omega_{\Lambda 0} \simeq 0.72$;
- $H_0 \simeq 70$ km s^{-1} Mpc^{-1}, with an error of 1-2%.

Briefly explain the *observational* evidence underpinning this model. Make sure you account for all the features listed above.

43. The Planck function can be written

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_{\mathrm{B}}T}\right) - 1},$$

where h is Planck's constant and k_{B} is Boltzmann's constant.

(a) Show that, if the universe expands by a factor a, this has the effect of decreasing the effective temperature of the radiation by the same factor (i.e. $T' = T/a$) while preserving the form of the blackbody distribution. Relate this to the observed spectrum of the cosmic microwave background.

(b) An early alternative to the expanding universe was the "tired light" model, in which light loses energy as it propagates through space, producing a redshift proportional to distance as observed in Hubble's law. Would this explanation of the cosmological redshift also preserve the blackbody law? Explain.

44. The abundances of the light nuclides ^{2}H, ^{4}He and ^{7}Li provide key evidence in support of a hot, dense early universe.

    (a) Briefly explain why these nuclides (together with ^{3}He), and *only* these nuclides, are produced in the early universe (and not, as was originally suggested, *all* heavy elements).

    (b) List the variables that influence the amount of ^{2}H and ^{4}He produced during Big Bang nucleosynthesis, and briefly explain *why* they have an effect.

    (c) How are these abundances determined observationally? What problems are encountered?

45. Type Ia supernovae (SNe Ia) are extensively used as cosmological probes.

    (a) Briefly explain the underlying physics of SNe Ia, and discuss why this makes them a better cosmological tool than Type II supernovae.

    (b) SNe Ia were originally thought to be genuine "standard candles", all reaching the same peak luminosity. Further work showed that this is not in fact the case. Why did this not make SNe Ia useless as cosmological probes? Did it introduce any problems at all?

    (c) SNe Ia are rare objects, occurring about once per Milky Way sized galaxy per century. Is this a problem with regard to their use in cosmology, and if so, in what respect?

    (d) SNe Ia were most famously used to investigate the z-dependence of H, providing the first unambiguous evidence for dark energy (and winning a Nobel prize for Perlmutter, Riess and Schmidt). Explain why they are particularly well suited for this task.

46. (a) Strong gravitational lenses can be used to measure H_0. Explain, with the aid of a diagram, the principles of gravitational lensing.

    (b) What are the advantages and disadvantages of using gravitational lenses in this way?
    [You may find it useful to look at SH Suyu et al., *ApJ* **711** (2010) 201 (arXiv astro-ph/0910.2773), to see what needs to be done to make a gravitational lens system yield a competitive value of H_0.]

47. It is generally assumed that the accelerated expansion of the universe is accounted for by a cosmological constant Λ with equation of state $P = -\mathcal{E}$. However, this is not the only option.

    (a) The equation of state shown above corresponds to a non-zero vacuum energy density. Explain why it is reasonable to *expect* that the vacuum would have a non-zero energy density, but why it is *not* reasonable to expect that its value would correspond to what we observe.

    (b) If the dark energy is not a vacuum energy but is something more exotic, its equation of state, $P = w\mathcal{E}$, might not have $w = -1$. Use the acceleration equation to derive the upper limit on w required for accelerated expansion.

    (c) A particle cosmologist proposes a dynamical model for dark energy with an equation of state $P = -0.9\mathcal{E}$.

        (i) Assuming that the best fit value of $\Omega_{\mathrm{m}0}$ for this model is still 0.28, at what redshift would the expansion of the universe switch from decelerating to accelerating?

        (ii) Suppose we make the approximation that we can neglect the matter density, i.e. treat the universe as having $\Omega_{\mathrm{DE},0} = 1$, where the dark energy equation of state is given above. Solve the Friedman equation for this model, obtaining a solution of the form $a(t) = (t/t_0)^n$, where your solution will give you expressions for t_0 and n.

        (iii) Use your solution to obtain an expression for the comoving proper distance of an object at redshift z. Comment on the horizon distance in this model.

3.3 Structure Growth and the Cosmic Microwave Background

48. By considering a sphere of radius R, mass M and density

$$\rho(t) = \bar{\rho}(t)\,(1 + \delta(t))\,,$$

where $\bar{\rho}(t)$ is the mean matter density of the universe at time t, show that the initial stages of structure growth can be described by the differential equation

$$\ddot{\delta} + 2H\dot{\delta} - \tfrac{3}{2}\Omega_{\rm m}\delta = 0. \qquad (10)$$

State any assumptions that you have made in deriving this expression.

49. (a) Using equation 10, show that in a universe dominated by radiation, density perturbations grow only logarithmically, and that in a Λ-dominated universe they do not grow at all.

    (b) By using a trial solution of the form $\delta = Ct^n$, show that in a matter-dominated universe overdensities grow $\propto a$.

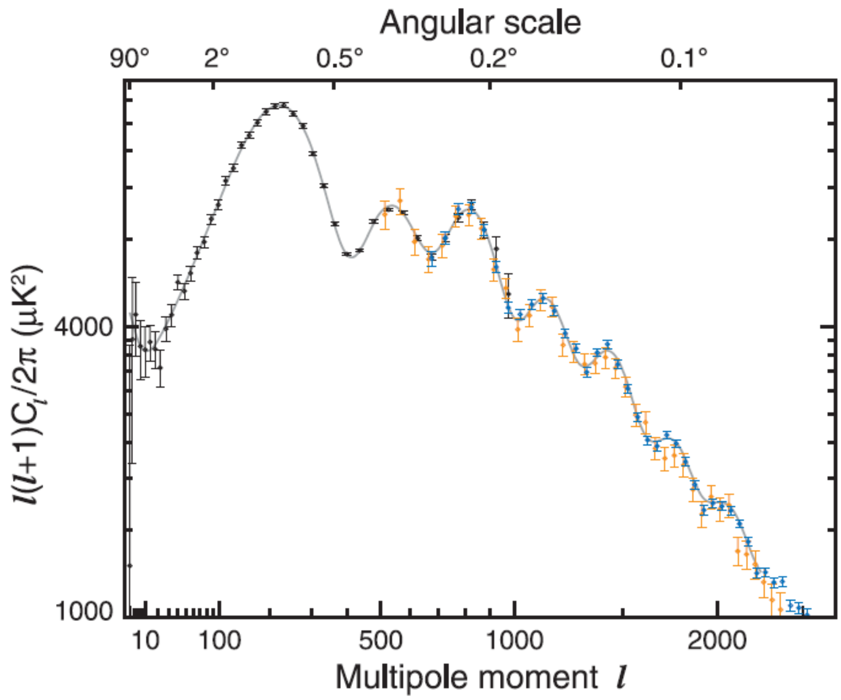
    (c) Explain why, despite the above result, overdensities in a universe in which all the matter is baryonic might *not* start to grow as soon as matter dominates over radiation.

    (d) Contrast the effects of **hot** and **cold** dark matter on structure formation.

50. (a) Explain why equation 10 cannot be used to describe the growth of structure from its initial stages until the present day. How, therefore, are theoretical predictions of large-scale structure produced?

    (b) Explain the relevance of (i) the cosmic microwave background and (ii) galaxy redshift surveys to the study of structure formation.

51. The image below shows the CMB power spectrum from the 9-year WMAP dataset (black), the South Pole Telescope (blue) and the Atacama Cosmology Telescope (orange).



(a) Explain how the sky map produced by the instruments is processed in order to generate this plot.

(b) From the point of view of cosmological parameter determination, what is the principal significance of (i) the position of the first peak and (ii) the ratio of the heights of odd and even peaks? In each case, explain the underlying physics.

(c) Despite the beautiful precision of these data, the WMAP team make use of non-CMB datasets, namely baryon acoustic oscillations (from galaxy surveys), H_0 measurements, and measurements using Type Ia supernovae. Explain why it is beneficial to include this additional information, and justify the choice of these particular measurements.

4 (Mostly) Numerical Calculations

52. Consider a matter-only universe ($\Omega_{\rm m0} = 1$).

    (a) At what redshift does the angular size distance $d_{\rm A}$ reach its maximum value?

    (b) At this redshift, and assuming $H_0 = 50$ km s^{-1} Mpc^{-1}, what is

       (i) the apparent magnitude of a Type Ia supernova with absolute magnitude −19.5;

       (ii) the linear separation of the two radio lobes of a radio galaxy, if their angular separation is $2''$?

    (c) With the same value of H_0, what is:

       (i) the age of the universe;

       (ii) the luminosity distance of a cluster whose look-back time is 10 Gyr;

       (iii) the angular diameter distance of a cluster whose luminosity distance is 3 Gpc?

53. For an open universe with $H_0 = 70$ km s^{-1} Mpc^{-1} and $\Omega_{\rm m0} = 0.21$, use equations 4 to determine:

    (i) the age of the universe;

    (ii) the look-back time to a supernova at redshift 1.4;

    (iii) the apparent magnitude of this supernova, if it is a Type Ia with peak absolute magnitude −19;

    (iv) the expected angular size of its host galaxy, if it is a typical large spiral with a linear diameter of 20 kpc.

54. Using equations 4 or the matter-only solution, as appropriate, calculate the angular diameter distance corresponding to a redshift of 1100 for $\Omega_{\rm m0} =$ (i) 0.10, (ii) 0.25, (iii) 0.50, (iv) 0.75 and (v) 1.0. Express your answer in units of c/H_0 (the Hubble distance).

    Comment on the implication of your results for the power spectrum of the cosmic microwave background.

    Note: remember that for a non-flat universe $d_{\rm A} \neq d_{\rm P}/(1+z)$*!*

55. (a) Before correction, the peak absolute magnitudes of Type Ia supernovae cover a range of about 0.8 magnitudes (from –18.7 to –19.5). What is the fractional error on the distance d of a supernova introduced by this spread?

    (b) If the systematic error on H_0 is $\pm 5\%$, how many supernovae would have to be included in your sample before the random error introduced by the spread was negligible compared to the systematic error (say, $< 2\%$)?

    (c) The Union 2.1 compilation (`http://supernova.lbl.gov/Union`) includes 580 individual Type Ia supernovae. Is it worth correcting their peak magnitudes? Justify your answer.

56. The latest WMAP results give, for the combined dataset, $H_0 = 69.7$ km s^{-1} Mpc^{-1}, $\Omega_{\mathrm{cdm},0} = 0.236$, $\Omega_{\mathrm{b}0} = 0.046$ and $z_{\mathrm{eq}} = 3280$, where $\Omega_{\mathrm{cdm},0}$ is the current dark matter density in units of the critical density, $\Omega_{\mathrm{b}0}$ the current baryon density, and z_{eq} the redshift of matter-radiation equality. The geometry of the universe is flat to high precision ($\Omega_{\mathrm{tot},0} = 1$).

    (a) Using equation 9, calculate the age of the universe.

    (b) What is the radiation density at the current time, $\Omega_{\mathrm{r}0}$?
    Compare this with the blackbody radiation density,

    $$\mathcal{E}_{\mathrm{BB}} = \frac{4\sigma}{c}T^4,$$

    where $T_{\mathrm{CMB}} = 2.725\,\mathrm{K}$, and explain any difference.

    (c) Find the redshift at which this universe switched from decelerating expansion to accelerating expansion, and the look-back time for this redshift.

    (d) Calculate the values of Ω_Λ, Ω_{cdm}, Ω_{b}, Ω_{r} and H at the time at which the CMB was emitted, corresponding to $z = 1090$.