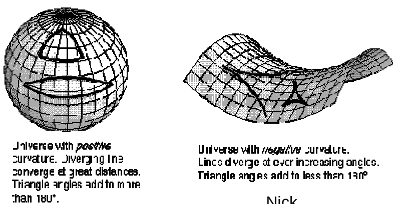
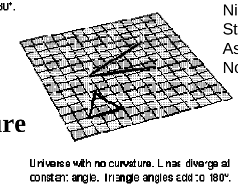
Curved spacetime and implications for cosmology

- **General Relativity implies spacetime is curved in the presence of matter**
  - **since universe contains matter, might expect overall curvature (as well as local “gravity wells”)**
  - **how does this affect measurements of large-scale distances?**
  - **what are the implications for cosmology?**

Curved spacetime

- **Two-dimensional curved space: surface of sphere**
  - **distance between (r,θ) and (r+dr,θ+dθ) given by**
  - **$ds^2 = dr^2 + R^2\sin^2(r/R)d\theta^2$**
  - **r = distance from pole**
    θ = angle from meridian
    R = radius of sphere
  - **positive curvature**
- **“Saddle” (negative curvature)**
  - **$ds^2 = dr^2 + R^2\sinh^2(r/R)d\theta^2$**
  - **(2D surface of constant negative curvature can’t really be constructed in 3D space)**





Nick Strobel’s Astronomy Notes

3D curved spacetime

- **Robertson-Walker metric**
  - $\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)[\mathrm{d}x^2/(1 - kx^2/R^2) + x^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2)]$
    - note sign change from our previous definition of $\mathrm{d}s^2$!
  - **$a(t)$ is an overall scale factor allowing for expansion or contraction ($a(t_0) \equiv 1$)**
  - **x is called a comoving coordinate (unchanged by overall expansion or contraction)**
  - **k defines sign of curvature ($k = \pm 1$ or 0), R is radius of curvature**
  - **path of photon has $\mathrm{d}s^2 = 0$, as before**

Implications for cosmology

- **comoving proper distance ($\mathrm{d}t = 0$) between origin and object at coordinate x:**

$$r = \int_0^x \frac{\mathrm{d}x}{\sqrt{1 - kx^2/R^2}}$$

  - **for $k = +1$ this gives $r = R\sin^{-1}(x/R)$, i.e. $r \le 2\pi R$**
    - finite but unbounded universe, cf. sphere
  - **for $k = -1$ we get $r = R\sinh^{-1}(x/R)$, and for $k = 0$, $r = x$**
    - infinite universe, cf. saddle
  - **for $x << R$ all values of k give $r \approx x$**
    - any spacetime looks flat on small enough scales
  - **this is independent of a**
    - it's a comoving distance

Implications for cosmology

- cosmological redshift: change variables in RW metric from x to r:
  - $ds^2 = -c^2dt^2 + a^2(t)[dr^2 + x(r)^2 d\Omega^2]$
  - for light $ds = 0$, so $c^2dt^2 = a^2(t)dr^2$, i.e. $c\,dt/a(t) = dr$ (assuming beam directed radially)
  - suppose wave crest emitted at time t_e and observed at t_o

$$c\int_{t_e}^{t_o}\frac{dt}{a(t)} = \int_0^r dr = r$$

first wave crest

$$c\int_{t_e+\lambda_e/c}^{t_o+\lambda_o/c}\frac{dt}{a(t)} = \int_0^r dr = r$$

next wave crest

Implications for cosmology

- Then $$c\int_{t_e}^{t_e+\lambda_e/c}\frac{dt}{a(t)} = c\int_{t_o}^{t_o+\lambda_o/c}\frac{dt}{a(t)}$$

  but if $\lambda << c/H_0$, $a(t)$ is almost constant over this integral, so we can write

$$\frac{c}{a(t_e)}\int_{t_e}^{t_e+\lambda_e/c}dt = \frac{c}{a(t_o)}\int_{t_o}^{t_o+\lambda_o/c}dt$$

i.e. $$\frac{\lambda_e}{a(t_e)} = \frac{\lambda_o}{a(t_o)}$$

Implications for cosmology

- **So expanding universe produces redshift *z*, where**

$$1 + z = \frac{a(t_o)}{a(t_e)} = \frac{1}{a(t_e)}$$

- **Note:**
  - ***z* can have any value from 0 to ∞**
  - ***z* is a measure of t_e**
  - **often <u>interpret</u> *z* using relativistic Doppler shift formula**

$$1 + z = \sqrt{\frac{c + v}{c - v}}$$

but note that this is misleading: the object is **<u>not</u>** changing its local coordinates

Implications for cosmology

- **Conclusions**
  - **in general relativity universe can be infinite (if *k* = −1 or 0) or finite but unbounded (if *k* = +1)**
  - **universe can expand or contract (if overall scale factor *a*(*t*) is not constant)**
  - **if universe expands or contracts, radiation emitted by a comoving source will appear redshifted or blueshifted respectively**