Solve the Friedmann equation for the case of a matter-dominated flat universe. [3]

The front cover equation gives $\dot{a}(t)^2 = \frac{8\pi G}{3c^2}\left(\frac{\varepsilon_{m0}}{a(t)}\right)$ after removal of irrelevant terms. [½]

Take the square root and collect terms to get $\sqrt{a}\,\mathrm{d}a = H_0\,\mathrm{d}t$, where for compactness I have written $H_0^2 = 8\pi G\varepsilon_{m0}/3c^2$. [½]

Then integrate this to obtain $\frac{2}{3}a^{3/2} = H_0 t$. [1]

Put in the condition that $a = 1$ at $t = t_0$ to get $a = (t/t_0)^{2/3}$ (and $t_0 = 2/3H_0$). [1]

Hence show that, in such a universe, the proper distance of an object at redshift z is

$$d_P = \frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right),$$

where H_0 is the present value of the Hubble parameter. [3]

Taking the expression for the proper distance from part (a) and substituting $a = (t/t_0)^{2/3}$, we get $d_P = ct_0^{2/3}\int_{t_e}^{t_0} t^{-2/3}\,\mathrm{d}t$. [½]

Doing the integral gives $d_P = 3ct_0\left(1 - (t_e/t_0)^{1/3}\right)$ [1]

and substituting $a = (t/t_0)^{2/3}$ makes this $d_P = 3ct_0(1 - a^{-1/2})$. [½]

If we now write $a = 1/(1+z)$ and $3t_0 = 2/H_0$ we get the required answer. [1]

In a flat, matter-dominated universe with H_0 = 50 km s^{-1} Mpc^{-1}, a quasar is observed at redshift z = 4.0.

(i) Calculate the proper distance of the quasar. [1]
Put the numbers in the equation above to get 21.6 Gly (6.6 Gpc). [1]

(ii) Calculate the time at which the light from the quasar was emitted, and hence the *look-back time* for the quasar. [2]
We know that $a = 1/(1+z) = 0.2$, and that $t_0 = 2/3H_0$ = 13.0 Gyr. [1]
Hence $t_e = t_0 a^{3/2}$ = 1.2 Gyr [½]
and so the lookback time is 13.0 – 1.2 = 11.8 Gyr. [½]