Topic 4 – Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

Today

- **Introduction to Fourier analysis**

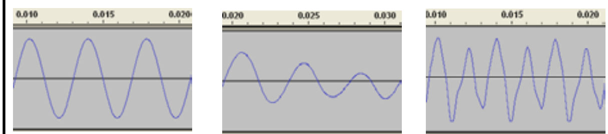
Fourier Series

Music is more than just pitch and amplitude………it's also about timbre.

The richness of a sound or note produced by a musical instrument is described in terms of a sum of a number of distinct frequencies called harmonics. The lowest frequency is called the fundamental frequency and the pitch it produces is used to name the note.

The balance of the amplitudes of the different harmonic frequencies is responsible for the characteristic sound of each instrument.

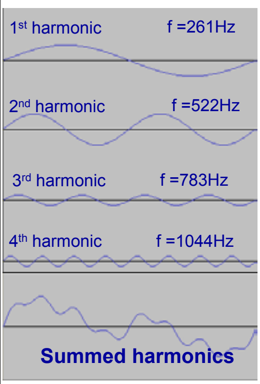
Each instrument below is playing a single note middle C (261Hz)



Signal generator | Classical guitar | Piano

What do you think is best instrument to demonstrate timbre? **Bagpipes**

A single note will contain different fractions of each harmonic



$y_1 = B_1 \sin\omega_1 t$

$y_2 = B_2 \sin\omega_2 t$

$y_3 = B_3 \sin\omega_3 t$

$y_4 = B_4 \sin\omega_4 t$

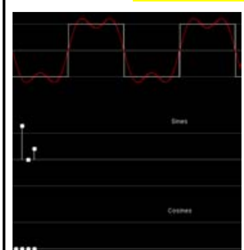
$y_{total} = B_1 \sin\omega_1 t + B_2 \sin\omega_2 t + ... = \sum_{n=1}^{\infty} B_n \sin\omega_n t$

The summed output repeats with the period of the 1st harmonic

Fourier Series

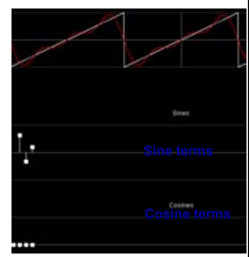
Fourier made the bold claim that any repeating pattern could therefore be represented by a summed series of cosine and sine terms with a term a_0 added to the sum to adjust the baseline if needed.

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t$$



b_n ω

a_n ω



http://www.univie.ac.at/future.media/moe/galerie/fourier/fourier.html http://www.falstad.com/fourier/

Fourier Series

Fourier said

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t$$

The pattern will repeat with period of the (n = 1) 1st harmonic frequency

$\omega = \omega_1$ so since $\omega_n = n\omega$ and $\omega = 2\pi f$ and $T = \frac{1}{f}$, then $\omega_n = \frac{2n\pi}{T}$

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}$$

where T is the period of the repeating shape

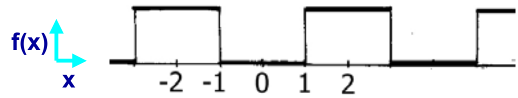
Fourier Series

Fourier said $f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi t}{T} + b_n \sin\frac{2n\pi t}{T}$

Or if the pattern repeats with period L where L is a distance

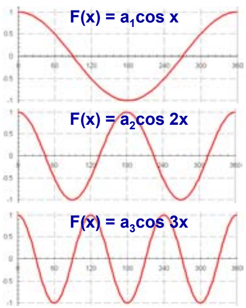
we can write $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$

Here period of the shape is L = 4



Fourier Series

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L}$$

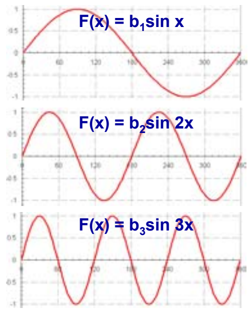


n = 1

For these cases L is taken to be 2π to simplify expressions

n = 2

n = 3

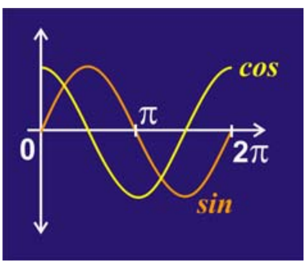


Need to find a_0, a_n and b_n

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L}$$

… some background work required ….

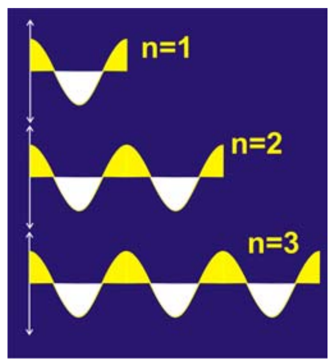
Integrating over 2π



$$\int_0^{2\pi}\cos x\ dx=0$$

$$\int_0^{2\pi}\sin x\ dx=0$$

Integrating over $2\pi n$



$$\int_0^{2\pi}\cos nx\ dx=0\quad for\ all\ \ n$$

Useful integrals

Try these integrals using the hints provided

$$\cos^2 x=\frac{1}{2}\cos 2x+\frac{1}{2}$$

$$\cos A\cos B=\frac{1}{2}\cos(A-B)+\frac{1}{2}\cos(A+B)$$

$$\int_0^{2\pi}\cos^2 nx\ \ dx \qquad =\int_0^{2\pi}\left(\frac{1}{2}\cos 2nx+\frac{1}{2}\right)\ dx=\left[\frac{1}{4n}\sin 2nx+\frac{x}{2}\right]_0^{2\pi}=\pi$$

$$\int_0^{2\pi}\cos nx\cos mx\ dx \qquad =\int_0^{2\pi}\frac{1}{2}\{\cos(n-m)x\}dx+\int_0^{2\pi}\frac{1}{2}\{\cos(n+m)x\}\ dx$$

$n\neq m$

$$=\left[\frac{\sin(n-m)x}{2(n-m)}\right]_0^{2\pi}+\left[\frac{\sin(n+m)x}{2(n+m)}\right]_0^{2\pi}=0$$

Useful Integrals summary

$$\int_0^{2\pi}\sin nx\ dx=0\quad for\ all\ \ n \qquad \int_0^{2\pi}\cos nx\ dx=0\quad for\ all\ \ n$$

Previous page

$$\int_0^{2\pi}\cos nx\cos mx\ dx=\int_0^{2\pi}\sin nx\sin mx\ dx=0\quad for\ all\ \ m\neq n$$

Previous page

$$\int_0^{2\pi}\sin^2 nx\ dx=\int_0^{2\pi}\cos^2 nx\ dx=\pi\quad for\ all\ \ n$$

Previous page

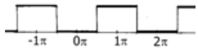
$$\int_0^{2\pi}\sin nx\cos mx\ dx=0\quad for\ all\ \ m\ and\ n$$

Remember odd x even function

Finding coefficients of the Fourier Series…a_0

Remember how the Fourier series can be written like this for a period L

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$



For simplicity let's make L = 2π so we can write…

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Take this equation and integrate both sides over a period

$$\int_0^{2\pi} f(x)dx = \frac{1}{2}a_0\int_0^{2\pi} dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\,dx + b_n\int_0^{2\pi}\sin nx\,dx\right]$$

Clearly on the RHS the only non-zero term is the a_0 term

$$\int_0^{2\pi} f(x)dx = \frac{1}{2}a_0\int_0^{2\pi} dx = \frac{1}{2}a_0(2\pi - 0) = \pi a_0$$

Hence we find $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$

Finding coefficients of the Fourier Series…a_n

Again let L = 2π so we can write… $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$

This time multiply both sides by ***cos(x)*** and integrate over a period

$$\int_0^{2\pi} f(x)\cos x dx = \frac{1}{2}a_0\int_0^{2\pi}\cos x dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos x\,dx + b_n\int_0^{2\pi}\sin nx\cos x\,dx\right]$$

On RHS, only the a_1 term survives as it is only term where n=1 (Orthogonality)

$$\int_0^{2\pi} f(x)\cos x dx = a_1\int_0^{2\pi}\cos x\cos x\,dx = a_1\int_0^{2\pi}\cos^2 x\,dx = a_1\pi$$

Hence we find $a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x dx$

Finding coefficients of the Fourier Series…a_n

How to find all coefficients a_n is clear.

To find a general expression multiply both sides of the Fourier series by **cos(*mx*)**, then integrate over a period:

$$\int_0^{2\pi} f(x)\cos mx dx = \frac{1}{2}a_0\int_0^{2\pi}\cos mx dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos mx\,dx + b_n\int_0^{2\pi}\sin nx\cos mx\,dx\right]$$

On the RHS, only the m = n term survives the integration

$$\int_0^{2\pi} f(x)\cos mx dx = a_m\int_0^{2\pi}\cos^2 mx\,dx = a_m\pi$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx dx$$

Coefficients of the Fourier Series…

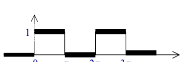
In a similar way, multiplying both sides of the Fourier series by **sin(*mx*)**, then integrating over a period we get:

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

Finding the coefficients of the Fourier Series



Step 1.
Write down the function $f(x)$ in terms of x.
What is the period?

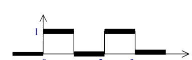
Step 2.
Use equation to find a_0? $a_0 = \frac{2}{L}\int_0^L f(x)dx$

Step 3.
Use equation to find a_n? $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx$

Step 4.
Use equation to find b_n? $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$

Example 4.1 - page 35

Find Fourier series to represent this repeat pattern.



1. Write down the function $f(x)$ in terms of x. What is period?

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$

Period is 2π

2. Use equation to find a_0?

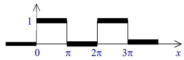
$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} 1dx + \frac{1}{\pi}\int_\pi^{2\pi} 0dx = \frac{1}{\pi}[x]_0^\pi = 1$$

3. Use equation to find a_n?

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\cos nx\,dx + \frac{1}{\pi}\int_\pi^{2\pi}(0)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^\pi = 0$$

Finding coefficients of the Fourier Series

Find Fourier series to represent this repeat pattern.



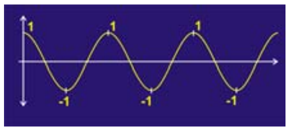
4. Use equation to find b_n?

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(0)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$b_n = \frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi} = \frac{-1}{\pi}\left(\frac{\cos n\pi}{n} - \frac{\cos 0}{n}\right) = \frac{-1}{\pi}\left(\frac{\cos n\pi}{n} - \frac{1}{n}\right)$$

$$b_n = \frac{-1}{\pi}\left(\frac{(-1)^n}{n} - \frac{1}{n}\right)$$



Step 5. Write out values of b_n for n = 1, 2, 3, 4, 5, ….

$$a_0 = 1$$
$$a_n = 0$$
$$b_n = \frac{-1}{\pi}\left(\frac{(-1)^n}{n} - \frac{1}{n}\right)$$

$$b_1 = \frac{1}{\pi}\left(1 - \frac{(-1)}{1}\right) = \frac{2}{\pi}$$
$$b_2 = \frac{1}{\pi}\left(\frac{1}{2} - \frac{(1)}{2}\right) = 0$$
$$b_3 = \frac{1}{\pi}\left(\frac{1}{3} - \frac{(-1)}{3}\right) = \frac{2}{3\pi}$$
$$b_4 = \frac{1}{\pi}\left(\frac{1}{4} - \frac{(1)}{4}\right) = 0$$
$$b_5 = \frac{1}{\pi}\left(\frac{1}{5} - \frac{(-1)}{5}\right) = \frac{2}{5\pi}$$

$$b_1 = \frac{1}{\pi}\left(1 - \frac{(-1)}{1}\right) = \frac{2}{\pi}$$
$$b_2 = \frac{1}{\pi}\left(\frac{1}{2} - \frac{(1)}{2}\right) = 0$$
$$b_3 = \frac{1}{\pi}\left(\frac{1}{3} - \frac{(-1)}{3}\right) = \frac{2}{3\pi}$$
$$b_4 = \frac{1}{\pi}\left(\frac{1}{4} - \frac{(1)}{4}\right) = 0$$
$$b_5 = \frac{1}{\pi}\left(\frac{1}{5} - \frac{(-1)}{5}\right) = \frac{2}{5\pi}$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

Step 6. Write out Fourier series

With period L = 2π

a_n, b_n in the generic form replaced with values for our example

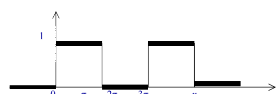
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{2\pi} + b_n \sin\frac{2n\pi x}{2\pi}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \ldots$$

Finding coefficients of the Fourier Series

So what does the Fourier series look like if we only use first few terms?

Use ***FourierChecker*** at ….



Fourier Series - QUIZ

1. What is $1+(-1)^n$ **when n = 3 ?** $1+(-1)^3 = 0$

2. What is $1+(-1)^n$ **when n = 52 ?** $1+(-1)^{52} = 2$

3. What is $1+(\cos 2\pi n)$ **when n = 1 ?** $1+(1) = 2$

4. What is $1+(\cos 2\pi n)$ **when n = 17 ?** $1+(1) = 2$

5. What is $1+(\cos 2\pi n)$ **when n = 52 ?** $1+(1) = 2$

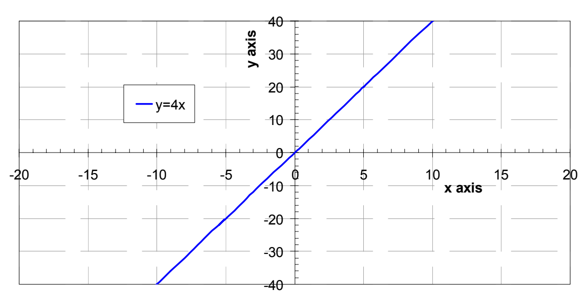
6. What is $1+(\cos \pi n)$ **when n = 1 ?** $1+(\cos\pi) = 1+(-1) = 0$

7. What is $1+(\cos \pi n)$ **when n = 4 ?** $1+(\cos 4\pi) = 1+(1) = 2$

Fourier Series - QUIZ

8. What is $I = \int_{-10}^{10} 4x\,dx$

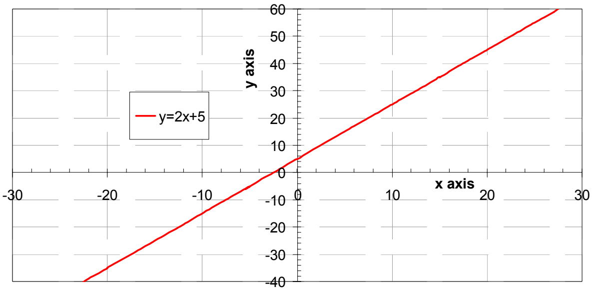
$$I = \int_{-10}^{10} 4x\,dx = \left[2x^2\right]_{-10}^{10} = 200 - 200$$



Fourier Series - QUIZ

9. Team A: What is $I = \int_0^{10} (2x+5)\, dx$?

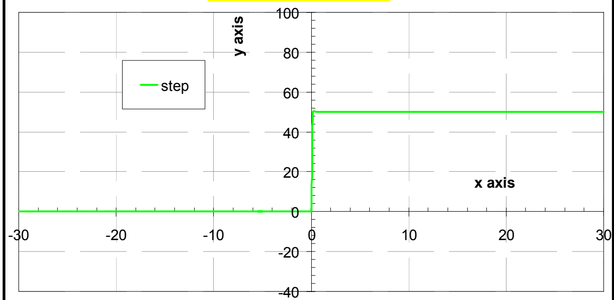
$$I = \int_0^{10} (2x+5)\, dx = \left[x^2 + 5x\right]_0^{10} = 150$$



Fourier Series - QUIZ

10. Team B: Describe the following step function in terms of *f(x)* and *x* ?

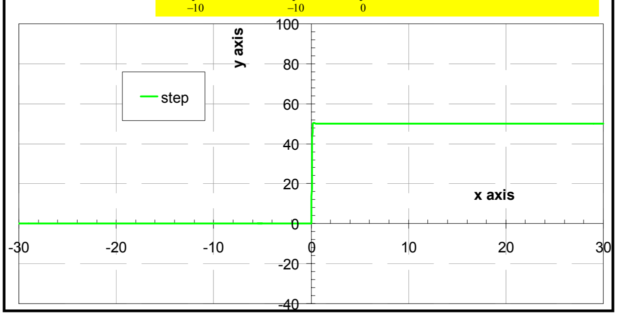
$$when \;\; x < 0 \quad f(x) = 0$$
$$when \;\; x > 0 \quad f(x) = 50$$



Fourier Series - QUIZ

11. Team A: What is $I = \int_{-10}^{10} f(x)\, dx$?

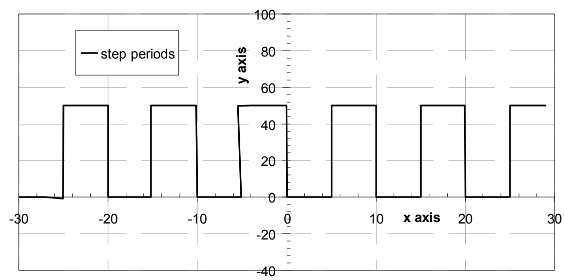
$$I = \int_{-10}^{10} f(x)\, dx = \int_{-10}^{0} 0\, dx + \int_0^{10} 50\, dx = [0x]_{-10}^{0} + [50x]_0^{10} = 500$$



Fourier Series - QUIZ

12. Team B: Describe the following step function over one period in terms of *f(x)* and *x* ?

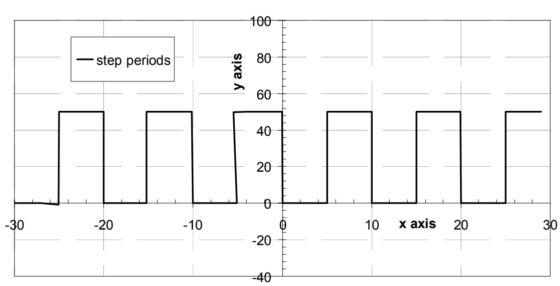
$$when \;\; 5 > x > 0 \quad f(x) = 0$$
$$when \;\; 10 > x \geq 5 \quad f(x) = 50$$



Fourier Series - QUIZ

13. Team A: What is the integral of *f(x)* over one period ?

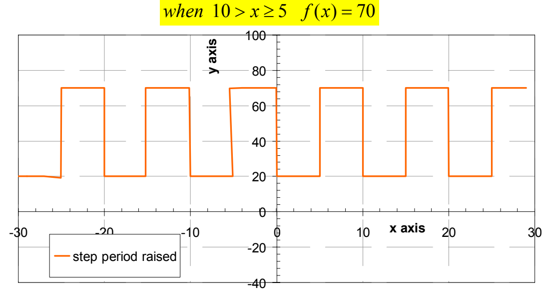
$$I = \int_0^{10} f(x)\, dx = \int_0^{5} 0\, dx + \int_5^{10} 50\, dx = [0x]_0^{5} + [50x]_5^{10} = 250$$



Fourier Series - QUIZ

14. Team B: Describe the following step function over one period in terms of *f(x)* and *x* ?

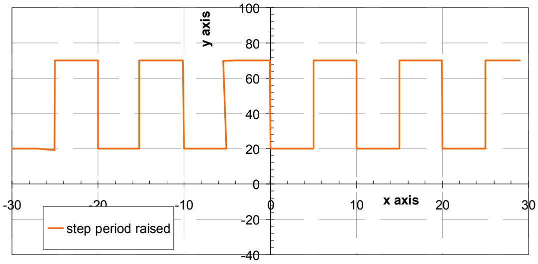
$$when \;\; 5 > x > 0 \quad f(x) = 20$$
$$when \;\; 10 > x \geq 5 \quad f(x) = 70$$



Fourier Series - QUIZ

15. Team A: What is the integral of *f(x)* over one period ?

$$I = \int_0^{10} f(x)\,dx = \int_0^{5} 20\,dx + \int_5^{10} 70\,dx = [20x]_0^5 + [70x]_5^{10} = 450$$

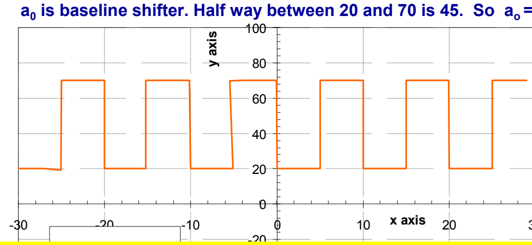


Fourier Series - QUIZ

Fourier series $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$

16. Team B: If we were to represent the function below as a Fourier series what could you say about the value of a_0 ?

a_0 is baseline shifter. Half way between 20 and 70 is 45. So $a_o = 90$



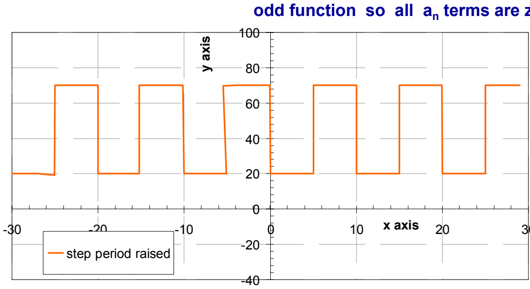
$$a_0 = \frac{2}{period}\int_0^{period} f(x)dx = \frac{2}{10}\int_0^5 20dx + \frac{2}{10}\int_5^{10} 70dx = \frac{1}{5}[20x]_0^5 + \frac{1}{5}[70x]_5^{10} = 20 + 70 = 90$$

Fourier Series - QUIZ

Fourier series $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$

17. Team A: If we were to represent the function below as a Fourier series what could you say about the values of the a_n terms ?

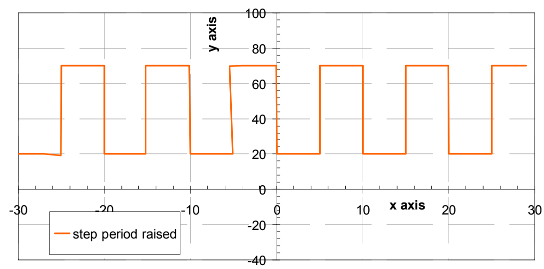
odd function so all a_n terms are zero



Fourier Series - QUIZ

Fourier series $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$

18. Team B: If we were to represent the function below as a Fourier series what could you say about the sign of the b_1 term ?

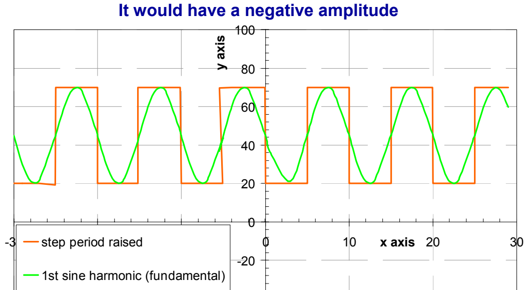


Fourier Series - QUIZ

Fourier series $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$

18. Team B: If we were to represent the function below as a Fourier series what could you say about the value of the b_1 term ?

It would have a negative amplitude



Lecture 7 – Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

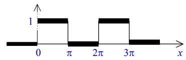
$$a_0 = \frac{2}{L}\int_0^L f(x)dx \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

Today

- More examples of Fourier series
- Describing pulses with Fourier series

Finding coefficients of the Fourier Series

Find Fourier series to represent this repeat pattern.



Steps to calculate coefficients of Fourier series

1. Write down the function *f(x)* in terms of *x*. What is period? $f(x)=\begin{cases}1 & 0<x<\pi\\ 0 & \pi<x<2\pi\end{cases}$

Period is 2π

2. Use equation to find a_0? $a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)dx=\frac{1}{\pi}\int_0^{\pi} 1dx+\frac{1}{\pi}\int_\pi^{2\pi} 0dx=\frac{1}{\pi}[x]_0^\pi=1$

3. Use equation to find a_n?

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}(1)\cos nx\,dx+\frac{1}{\pi}\int_\pi^{2\pi}(0)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}\cos nx\,dx=\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^\pi=0$$

4. Use equation to find b_n?

$$b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi}(1)\sin nx\,dx+\frac{1}{\pi}\int_\pi^{2\pi}(0)\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi}\sin nx\,dx=\frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^\pi$$

$$b_n=\frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^\pi=\frac{-1}{\pi}\left(\frac{\cos n\pi}{n}-\frac{\cos 0}{n}\right)=\frac{-1}{\pi}\left(\frac{\cos n\pi}{n}-\frac{1}{n}\right)$$

Finding coefficients of the Fourier Series

4. Use equation to find b_n? $b_n=\frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^\pi=\frac{-1}{\pi}\left(\frac{\cos n\pi}{n}-\frac{\cos 0}{n}\right)=\frac{1}{\pi}\left(\frac{1}{n}-\frac{\cos n\pi}{n}\right)$

5. Write out values of b_n for n = 1, 2, 3, 4, 5, ….

$b_1=\frac{1}{\pi}\left(\frac{1}{1}-\frac{\cos 1\pi}{1}\right)=\frac{1}{\pi}\left(1-\frac{(-1)}{1}\right)=\frac{2}{\pi}$ $b_2=\frac{1}{\pi}\left(\frac{1}{2}-\frac{\cos 2\pi}{2}\right)=\frac{1}{\pi}\left(\frac{1}{2}-\frac{(1)}{2}\right)=0$

$b_3=\frac{1}{\pi}\left(\frac{1}{3}-\frac{\cos 3\pi}{3}\right)=\frac{1}{\pi}\left(\frac{1}{3}-\frac{(-1)}{3}\right)=\frac{2}{3\pi}$ $b_4=\frac{1}{\pi}\left(\frac{1}{4}-\frac{\cos 4\pi}{4}\right)=\frac{1}{\pi}\left(\frac{1}{4}-\frac{(1)}{4}\right)=0$

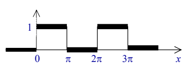
$b_5=\frac{1}{\pi}\left(\frac{1}{5}-\frac{\cos 5\pi}{5}\right)=\frac{1}{\pi}\left(\frac{1}{5}-\frac{(-1)}{5}\right)=\frac{2}{5\pi}$

6. Write out Fourier series $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L}$

with period L, a_n, b_n in the generic form replaced with values for our example

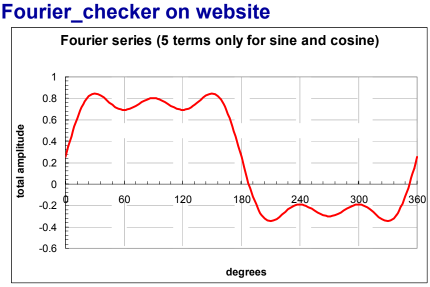
$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{2\pi}+b_n\sin\frac{2n\pi x}{2\pi}=\frac{1}{2}+\frac{2}{\pi}\sin x+\frac{2}{3\pi}\sin 3x+\frac{2}{5\pi}\sin 5x+\ldots$$

So what does this Fourier series look like if we only use first few terms?



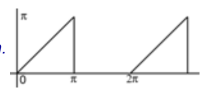
$$f(x)=\frac{1}{2}+\frac{2}{\pi}\sin x+\frac{2}{3\pi}\sin 3x+\frac{2}{5\pi}\sin 5x+\ldots$$

Use Fourier_checker on website



Finding coefficients of the Fourier Series

Find Fourier series to represent this repeat pattern.



Steps to calculate coefficients of Fourier series

1. Write down the function *f(x)* in terms of *x*. What is period?

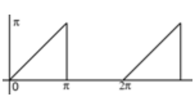
$f(x)=\begin{cases}x & 0<x<\pi\\ 0 & \pi\le x<2\pi\end{cases}$ Period is 2π

2. Use equation to find a_0?

$$a_0=\frac{2}{L}\int_0^L f(x)dx=\frac{2}{2\pi}\int_0^{2\pi} f(x)dx=\frac{1}{\pi}\int_0^{\pi} xdx+\frac{1}{\pi}\int_\pi^{2\pi} 0dx=\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^\pi=\frac{\pi}{2}$$

Left side - find coefficients a_n **Right side - find coefficients b_n**

Finding coefficients of the Fourier Series



$f(x)=\begin{cases}x & 0<x<\pi\\ 0 & \pi\le x<2\pi\end{cases}$

First 5 terms (*n=1 to 5*)

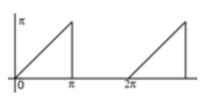
Left side - find coefficients a_n

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx=\frac{1}{\pi}\int_0^L f(x)\cos nx\,dx$$

Right side - find coefficients b_n

$$b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx=\frac{1}{\pi}\int_0^L f(x)\sin nx\,dx$$

Attendance sheet



$f(x)=\begin{cases}x & 0<x<\pi\\ 0 & \pi\le x<2\pi\end{cases}$ $a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

Period is 2π

Coefficients a_n $a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx+\frac{1}{\pi}\int_\pi^{2\pi} 0\cos nx\,dx$

Integrate by parts $\int udv=uv-\int vdu$ $u=x$ $dv=\cos nxdx$

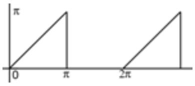
$du=dx$ $v=\int\cos nxdx=\frac{1}{n}\sin nx$

a

$$a_n=\left[\frac{x}{\pi n}\sin nx\right]_0^\pi-\frac{1}{\pi}\int_0^\pi\frac{1}{n}\sin nxdx$$

$$a_n=\left[\frac{x}{\pi n}\sin nx\right]_0^\pi+\left[\frac{1}{\pi n^2}\cos nx\right]_0^\pi$$

$$a_n=\left(\frac{1}{n}\sin n\pi+\frac{1}{\pi n^2}\cos n\pi\right)-\left(0+\frac{1}{\pi n^2}\right)$$



$f(x) = \begin{cases} x & 0 < x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$ $\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

Period is 2π

Coefficients a_n

a

$$a_n = \left(\frac{1}{n}\sin n\pi + \frac{1}{\pi n^2}\cos n\pi\right) - \left(\frac{1}{\pi n^2}\right)$$

$$a_n = \left(\frac{1}{\pi n^2}(-1)^n\right) - \left(\frac{1}{\pi n^2}\right) = \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

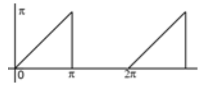
$a_1 = \left(-\frac{1}{\pi}\right) - \left(\frac{1}{\pi}\right) = -\frac{2}{\pi}$

$a_2 = \left(\frac{1}{4\pi}\right) - \left(\frac{1}{4\pi}\right) = 0$

$a_3 = \left(-\frac{1}{9\pi}\right) - \left(\frac{1}{9\pi}\right) = -\frac{2}{9\pi}$

$a_4 = \left(\frac{1}{16\pi}\right) - \left(\frac{1}{16\pi}\right) = 0$

$a_5 = \left(-\frac{1}{25\pi}\right) - \left(\frac{1}{25\pi}\right) = -\frac{2}{25\pi}$



$f(x) = \begin{cases} x & 0 < x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$ $\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Period is 2π

Coefficients b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\sin nx\,dx$$

Integrate by parts $\int u\,dv = uv - \int v\,du$

$u = x \qquad dv = \sin nx\,dx$

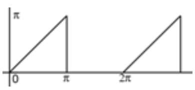
$du = dx \qquad v = \int \sin nx\,dx = -\frac{1}{n}\cos nx$

b

$$b_n = -\frac{1}{\pi}\left[\frac{x}{n}\cos nx\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\frac{1}{n}\cos nx\,dx$$

$$b_n = \left[-\frac{x}{\pi n}\cos nx\right]_0^{\pi} + \left[\frac{1}{\pi n^2}\sin nx\right]_0^{\pi}$$

$$b_n = \left(-\frac{1}{n}\cos n\pi + \frac{1}{\pi n^2}\sin n\pi\right)$$



$f(x) = \begin{cases} x & 0 < x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$ $\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$

Period is 2π

Coefficients b_n

$$b_n = \left(-\frac{1}{n}\cos n\pi + \frac{1}{\pi n^2}\sin n\pi\right)$$

$$b_n = -\frac{1}{n}(-1)^n$$

b

$b_1 = 1$

$b_2 = -\frac{1}{2}$

$b_3 = \frac{1}{3}$

$b_4 = -\frac{1}{4}$

$b_5 = \frac{1}{5}$

$a_0 = \frac{\pi}{2}$

$a_1 = -\frac{2}{\pi}$ $\quad a_4 = 0$

$a_2 = 0$ $\quad a_5 = -\frac{2}{25\pi}$

$a_3 = -\frac{2}{9\pi}$

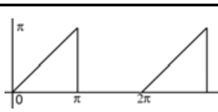
$b_1 = 1$ $\quad b_4 = -\frac{1}{4}$

$b_2 = -\frac{1}{2}$ $\quad b_5 = \frac{1}{5}$

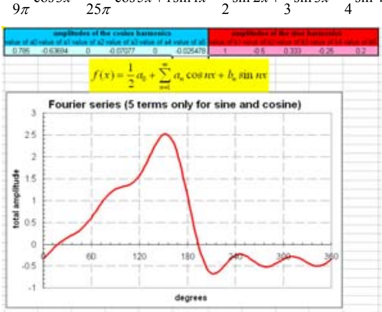
$b_3 = \frac{1}{3}$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\cos 1x - \frac{2}{9\pi}\cos 3x - \frac{2}{25\pi}\cos 5x + 1\sin 1x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x + ...$$

Check our Fourier series using Fourier_checker.xls

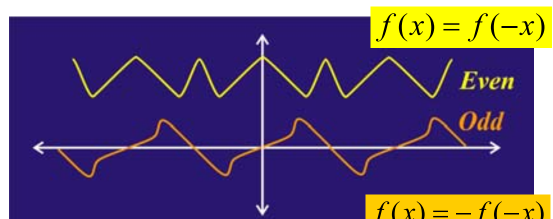


$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\cos 1x - \frac{2}{9\pi}\cos 3x - \frac{2}{25\pi}\cos 5x + 1\sin 1x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x + ...$$



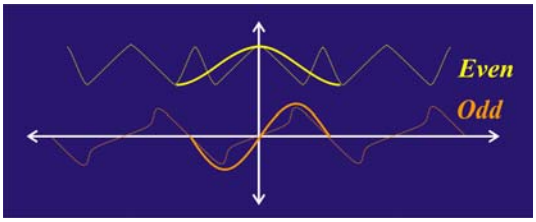
Fourier Series of even and odd repeating functions

$f(x) = f(-x)$



$f(x) = -f(-x)$

Fourier Series of even and odd repeating functions



Only *sine* terms required to define an **odd function**

Only *cosine* terms required to define an **even function**

Only an even function can have an offset.

Fourier Series applied to pulses

$$f(x)=\begin{cases} =0 \ \ for\ x<0 \\ \neq 0\ for\ 0\le x\le d \\ =0 \ \ for\ x>d \end{cases}$$

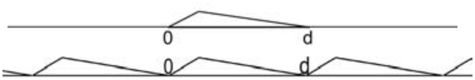
Laser light pulse

Initial displacement of a guitar string

Electronic wavefunction of a molecule

Can Fourier series be used to represent a single pulse event rather than an infinitely repeating pattern ?

Yes. It's fine if you have a single pulse occurring between 0 and d to pretend that it is part of an infinitely repeating pattern. All you have to do then is work out the Fourier series for the infinite pattern and say that the pulse is represented by that function just between 0 and d



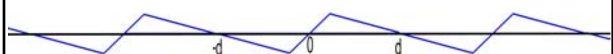
becomes

but only look between 0 and d

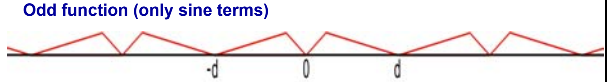
This approach is fine but it leads to a lot of work in the integration stage.

There is a better way….

If the only condition is that the pretend function be periodic, and since we know that even functions contain only cosine terms and odd functions only sine terms, why don't we draw it either like this or this?



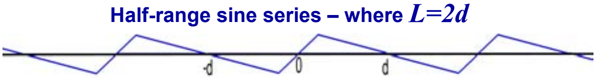
Odd function (only sine terms)



Even function (only cosine terms)

What is period of repeating pattern now?

Half-range sine series – where $L=2d$



We saw earlier that for a function with period *L* the Fourier series is:-

$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L}$ where $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx$ $\quad b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$

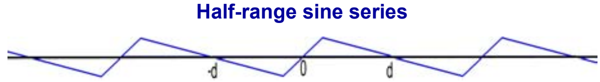
In the half range case we have a function of **period 2*d*** which is odd and so contains only sine terms, so the formulae become:-

$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{2n\pi x}{(2d)}=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{d}$$ where

$$b_n=\frac{2}{2d}\int_{-d}^{d}f(x)\sin\frac{2n\pi x}{(2d)}dx=\frac{1}{d}\int_{-d}^{d}f(x)\sin\frac{n\pi x}{d}dx$$

Remember, this is all to simplify the Fourier series. We're still only allowed to look at the function between x = 0 and x = d

Half-range sine series



$$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{d} \qquad b_n=\frac{1}{d}\int_{-d}^{d}f(x)\sin\frac{n\pi x}{d}dx$$

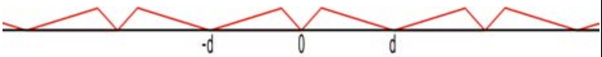
$$f(x)=ODD$$
$$\sin x=ODD$$
$$ODD\times ODD=EVEN$$

$$\int_{-d}^{d}EVEN=2\int_0^d EVEN$$

$$b_n=\frac{2}{d}\int_0^d f(x)\sin\frac{n\pi x}{d}dx$$

Fourier Series applied to pulses

Half-range cosine series



Again, for a function with period *L* the Fourier series is:-

$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L}$ where $a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx$ $b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$

Again we have a function of period *2d* but this time it is even and so contains only cosine terms, so the formulae become:-

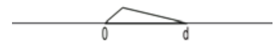
$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{(2d)}=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{d}$$ where

$$a_n=\frac{2}{2d}\int_{-d}^{d}f(x)\cos\frac{2n\pi x}{(2d)}dx=\frac{1}{d}\int_{-d}^{d}f(x)\cos\frac{n\pi x}{d}dx=\frac{2}{d}\int_0^d f(x)\cos\frac{n\pi x}{d}dx$$

$f(x)=EVEN$
$\cos x=EVEN$ $\int_{-d}^{d}EVEN=2\int_0^d EVEN$

Half-range sine and cosine series

The Fourier series for a pulse such as



can be written as either a half range sine or cosine series. However the series is only valid between 0 and d

Half range sine series	$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{d}$	where	$b_n=\frac{2}{d}\int_0^d f(x)\sin\frac{n\pi x}{d}dx$
Half range cosine series	$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{d}$	where	$a_n=\frac{2}{d}\int_0^d f(x)\cos\frac{n\pi x}{d}dx$

Let's do an example to demonstrate this………

Half Range Fourier Series

Find Half Range ***Sine*** Series which represents the displacement ***f(x),*** between *x* = 0 and 6, of the pulse shown to the right

The pulse is defined as $f(x)=x \quad for\ 0<x\leq 6$ with a length d = 6

So $b_n=\frac{2}{d}\int_0^d f(x)\sin\frac{n\pi x}{d}dx=\frac{2}{6}\int_0^6 x\sin\frac{n\pi x}{6}dx$ Integrate by parts $\int udv=uv-\int vdu$

$so\ set\ u=x\ and\ \sin\frac{n\pi x}{6}dx=dv$ $v=\int\sin\frac{n\pi x}{6}dx=-\frac{6}{n\pi}\cos\frac{n\pi x}{6}\quad as\quad du=dx$

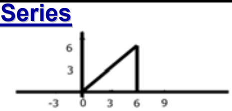
$$b_n=\frac{1}{3}\left(\left[-\frac{6x}{n\pi}\cos\frac{n\pi x}{6}\right]_0^6+\int_0^6\frac{6}{n\pi}\cos\frac{n\pi x}{6}\right)=\frac{1}{3}\left(\left[-\frac{6x}{n\pi}\cos\frac{n\pi x}{6}\right]_0^6+\left[\frac{36}{n^2\pi^2}\sin\frac{n\pi x}{6}\right]_0^6\right)$$

$$b_n=\frac{1}{3}\left(\left[-\frac{36}{n\pi}\cos n\pi\right]+\left[\frac{36}{n^2\pi^2}\sin n\pi\right]\right)=\frac{12}{n^2\pi^2}\sin n\pi-\frac{12}{n\pi}\cos n\pi$$

n=1	n=2	n=3	n=4	n=5
$b_1=\frac{12}{\pi}$	$b_2=-\frac{12}{2\pi}=-\frac{6}{\pi}$	$b_3=\frac{12}{3\pi}=\frac{4}{\pi}$	$b_4=-\frac{12}{4\pi}=-\frac{3}{\pi}$	$b_5=\frac{12}{5\pi}$

Half Range Fourier Series

Find Half Range Sine Series which represents the displacement f(x), between x = 0 and 6, of the pulse shown to the right



Half range sine series	$f(x)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{d}$	where	$b_n=\frac{2}{d}\int_0^d f(x)\sin\frac{n\pi x}{d}dx$

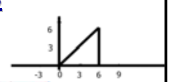
$$f(x)=\frac{12}{\pi}\sin\frac{\pi x}{6}-\frac{6}{\pi}\sin\frac{\pi x}{3}+\frac{4}{\pi}\sin\frac{\pi x}{2}-\frac{3}{\pi}\sin\frac{2\pi x}{3}+\frac{12}{5\pi}\sin\frac{5\pi x}{6}+.....$$

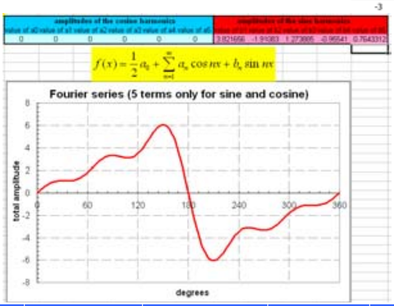
Can we check this on Fourier_checker.xls

n=1	n=2	n=3	n=4	n=5
$b_1=\frac{12}{\pi}$	$b_2=-\frac{12}{2\pi}=-\frac{6}{\pi}$	$b_3=\frac{12}{3\pi}=\frac{4}{\pi}$	$b_4=-\frac{12}{4\pi}=-\frac{3}{\pi}$	$b_5=\frac{12}{5\pi}$

Half Range Fourier Series

$$f(x)=\frac{12}{\pi}\sin\frac{\pi x}{6}-\frac{6}{\pi}\sin\frac{\pi x}{3}+\frac{4}{\pi}\sin\frac{\pi x}{2}-\frac{3}{\pi}\sin\frac{2\pi x}{3}+\frac{12}{5\pi}\sin\frac{5\pi x}{6}+.....$$





n=1	n=2	n=3	n=4	n=5
$b_1=\frac{12}{\pi}$	$b_2=-\frac{12}{2\pi}=-\frac{6}{\pi}$	$b_3=\frac{12}{3\pi}=\frac{4}{\pi}$	$b_4=-\frac{12}{4\pi}=-\frac{3}{\pi}$	$b_5=\frac{12}{5\pi}$

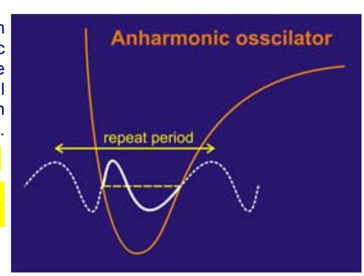
Fourier Series applied to pulses

Why is this useful?

In Quantum you have seen that there exist specific solutions to the wave equation within a potential well subject to the given boundary conditions.

$\Psi_n=0\ \ when\ \ x=0\ and\ x=d$

$\Psi(x)=\sum_{n=1}^{\infty}\Psi_n(x)=\sum_{n=1}^{\infty}B_n\sin\frac{n\pi x}{d}$



Given a complicated general solution we can deconvolve into harmonic terms

Lecture 7 – Summary

- **Practice questions – online at** http://www.hep.shef.ac.uk/phy226/unit1/phy226T1to4.htm
- **Normal series**
- **Even and odd functions**
- **Pulses**
- **Repeat period of 2π and other arbitrary repeat periods**

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

Lecture 8 – Summary

- **Practice questions – online at** http://www.hep.shef.ac.uk/phy226/unit1/phy226T1to4.htm
- **Complex Fourier series**
- **Parseval's theorum**
- **Revision & Practice**

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

Complex Fourier Series

In many areas of physics, especially Quantum mechanics, it is more convenient to consider waves written in their complex form

Remember:-

$$\cos\frac{2n\pi x}{L} = \cos nx = \frac{1}{2}(e^{inx} + e^{-inx}) \qquad \sin\frac{2n\pi x}{L} = \sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$$

The complex form of the Fourier series can be derived by assuming a solution of the form $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and then multiplying both sides by e^{-imx} and integrating over a period:

$$\int_0^{2\pi} f(x)e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{inx}e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{i(n-m)x}dx$$

For $n \neq m$ integral vanishes. For *n=m* integral = 2π. So $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$

Complex Fourier Series

The complex form of the Fourier series can be derived by assuming a solution of the form $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and then multiplying both sides by e^{-imx} and integrating over a period:

$$\int_0^{2\pi} f(x)e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{inx}e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{i(n-m)x}dx$$

$$\int_0^{2\pi} e^{i(n-m)x}dx = \int_0^{2\pi}\cos(n-m)x\,dx + i\int_0^{2\pi}\sin(n-m)x\,dx$$

For $n \neq m$ integral vanishes. For *n=m* integral = 2π. So $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$

Complex Fourier Series

Therefore for a period of 2π the complex Fourier series is given as:-

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$$

The more general expression for a function *f(x)* with **period *L*** can be expressed as:-

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi inx}{L}} \quad \text{where} \quad c_n = \frac{1}{L}\int_0^L f(x)e^{-\frac{2\pi inx}{L}}dx$$

Let's try an example from the notes …..

Complex Fourier Series example

Example 4.5

Find the complex Fourier series for *f(x)* = *x* in the range -2 < *x* < 2 if the repeat period is 4.

$$c_n = \frac{1}{L}\int_0^L f(x)e^{\frac{-2\pi inx}{L}}dx$$ and period is 4, so we write $$c_n = \frac{1}{4}\int_{-2}^{2} xe^{\frac{-\pi inx}{2}}dx$$

Integration by parts $\int u\,dv = uv - \int v\,du$ $\quad with\ u = x\ \ and\ \ dv = e^{\frac{-\pi inx}{2}}dx$

$$So\ \ du = dx\ \ and\ \ v = \frac{-2}{\pi in}e^{\frac{-\pi inx}{2}}$$

$$c_n = \frac{1}{4}\left[\frac{-2x}{\pi in}e^{\frac{-\pi inx}{2}} + \int\frac{2}{\pi in}e^{\frac{-\pi inx}{2}}dx\right]_{-2}^{2} = \frac{1}{4}\left[\frac{-2x}{\pi in}e^{\frac{-\pi inx}{2}} - \frac{4}{\pi^2 i^2 n^2}e^{\frac{-\pi inx}{2}}\right]_{-2}^{2} = \left[\frac{-x}{2\pi in}e^{\frac{-\pi inx}{2}} + \frac{1}{\pi^2 n^2}e^{\frac{-\pi inx}{2}}\right]_{-2}^{2}$$

$$C_n = \left[\frac{-1}{\pi in}e^{-\pi in} + \frac{1}{\pi^2 n^2}e^{-\pi in}\right] - \left[\frac{1}{\pi in}e^{\pi in} + \frac{1}{\pi^2 n^2}e^{\pi in}\right] = \frac{-1}{\pi in}\left(e^{-\pi in} + e^{\pi in}\right) + \frac{1}{\pi^2 n^2}\left(e^{-\pi in} - e^{\pi in}\right)$$

Complex Fourier Series example

Example 6
Find the complex Fourier series for $f(x) = x$ in the range $-2 < x < 2$ if the repeat period is 4.

Since $\frac{-1}{i}\times\frac{i}{i}=i$ then $C_n=\frac{i}{\pi n}\left(e^{-\pi i n}+e^{\pi i n}\right)+\frac{1}{\pi^2 n^2}\left(e^{-\pi i n}-e^{\pi i n}\right)$

We want to find actual values for C_n so it would be helpful to convert expression for C_n into sine and cosine terms using the standard expressions:-

$$\cos n\pi=\frac{1}{2}\left(e^{-\pi i n}+e^{\pi i n}\right) \qquad \sin n\pi=\frac{-1}{2i}\left(e^{-\pi i n}-e^{\pi i n}\right)$$

So we can write…

$C_n=\frac{2i}{\pi n}\cos n\pi-\frac{2i}{\pi^2 n^2}\sin n\pi=\frac{2i}{\pi n}\cos n\pi$ so $C_n=\frac{2i}{\pi n}\cos n\pi=\frac{2i}{\pi n}(-1)^n$ and since

$$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{2\pi i n x}{L}} \quad \text{so} \quad f(x)=\sum_{n=-\infty}^{\infty}\frac{2i}{\pi n}(-1)^n e^{\frac{\pi i n x}{2}}$$

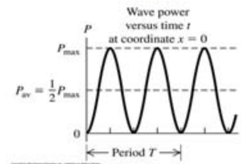
Parseval's Theorem applied to Fourier Series

The energy in a vibrating string or an electrical signal is proportional to the square of the amplitude of the wave

Sinusoidal Wave on a string

$$P_{\max}=\sqrt{\mu T}\,\omega^2 A^2$$

$$P_{ave}=\frac{1}{2}\sqrt{\mu T}\,\omega^2 A^2$$



NB, for all MECHANICAL waves, $P_{ave} \propto A^2\omega^2$

Parseval's Theorem applied to Fourier Series

Consider again the standard Fourier series with a period taken for simplicity as 2π …. $f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$

Square both sides then integrate over a period:

$$\int_0^{2\pi}[f(x)]^2dx=\int_0^{2\pi}\left[\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx\right]^2dx$$

The RHS will give both squared terms and cross term. When we integrate, *all* the cross terms will vanish. All the squares of the cosines and sines integrate to give π (half the period). Hence:-

$$\int_0^{2\pi}[f(x)]^2dx=\pi\frac{a_0^{\,2}}{2}+\pi\sum_{n=1}^{\infty}[a_n^{\,2}+b_n^{\,2}]$$

Hence Parseval's theorem tells us that the total energy in a vibrating system is equal to the sum of the energies in the individual modes.

Practice and revision

Try online questions 2 & 7

Fourier Series in movies!!!!

In music if the fundamental frequency of a note is 100 Hz

Then the harmonics are at 200 Hz, 300 Hz, 400 Hz, 500 Hz, 600 Hz

But the octaves are at 200 Hz, 400 Hz, 800 Hz, 1600 Hz, 3200 Hz

If you wanted to explain the Fourier series to an alien you'd probably pick notes that showed you understood this ……

So as a greeting why not try **G, A, F, $F_{\text{octave lower}}$, C**

G	A	F	F (octave lower)	C
392Hz	440Hz	349Hz	174Hz	261Hz

Why are these notes special ....We'll take the 1st harmonic as F_{bottom}=21.8Hz

G	A	F	F (octave lower)	C
18th harmonic	**20th harmonic**	**4th octave (16th harmonic)**	**8th harmonic**	**12th harmonic**

$$f(t)_{total}=B_{18}\sin(2\pi f_{18}t)+B_{20}\sin(2\pi f_{20}t)+B_{16}\sin(2\pi f_{16}t)+B_8\sin(2\pi f_8 t)+B_{12}\sin(2\pi f_{12}t)$$

http://uk.youtube.com/watch?v=tUcOaGawIW0