## 1st problem class – this week
## To be handed in next week

**Also remember the online problems**
**http://www.hep.shef.ac.uk/phy226**

## Check the fire exits – There may well be a fire drill today

# Topic 3

## Ordinary Differential equations

Purpose of lecture 3:

- Solve 1st order ODE
- Solve 2nd order homogeneous ODE

## Defining terms I

**1st order**
means the highest differential term is $\frac{dx}{dt}$

**2nd order**
means the highest differential term is $\frac{d^2x}{dt^2}$

**Ordinary**
means solutions are functions of one variable $x(t)$

**Homogeneous**
means that $f(t) = 0$ $\quad a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = f(t)$

## Defining terms II

**The General solution** is the broadest most long-winded version of the solution

$$N(t) = Ae^{mt}$$

**The Particular solution** is the result of applying boundary conditions to the general solution

$$N(t) = 5e^{-2t}$$

**Today we solve 1st order and 2nd order homogeneous ordinary differential equations…**

## 1st order homogeneous ODE

e.g. radioactive decay $\frac{dN(t)}{dt} = -\lambda N(t)$

$$\int \frac{dN}{N} = -\lambda \int dt$$

**1st method: Separation of variables**

**gives** $\ln N = -\lambda t + c$

$$N = e^{-\lambda t + c} = e^{-\lambda t} e^{c} = Ae^{-\lambda t}$$

## 1st order homogeneous ODE

e.g. radioactive decay $\frac{dN(t)}{dt} = -\lambda N(t)$

**2nd method: Trial solution**

Guess trial solution looks like $N(t) = Ae^{mt}$

Substitute the trial solution into the ODE

$$\frac{dN(t)}{dt} = Ame^{mt} = mN(t)$$

Comparison shows that $m = -\lambda$ **so write** $N(t) = Ae^{-\lambda t}$

# 1st order homogeneous ODE

**The general solution** $N(t) = Ae^{-\lambda t}$

**The particular solution is found by applying boundary conditions**

**e.g. At t=0 there are 100 nuclei; but at t=20 there were only 50 left**

**So we can write** $N(0) = 100 = Ae^{-\lambda 0}$ **so** $100 = A$

**The other boundary conditions tell us that** $N(20) = 50 = 100e^{-20\lambda}$

**So we can write** $\frac{50}{100} = e^{-20\lambda}$ **so** $\ln(0.5) = -20\lambda$ **so** $\lambda = 0.035$

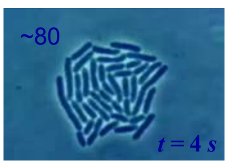
**The particular solution is therefore** $N(t) = 100e^{-0.035t}$

# Bacterial growth



http://www.youtube.com/watch?v=gEwzDydciWc

# Bacterial growth


# Bacterial growth

Boundary conditions

$t = 0\ s$ N = 4

$t = 4\ s$ N = 80

What does the general solution look like?

Find the particular solution

# Bacterial growth

Boundary conditions

$t = 0\ s$ N = 4

$t = 4\ s$ N = 80

# <u>Trial solution</u> is a powerful method of solving differential equations

## 2nd order homogeneous ODE

**Solving** $a\dfrac{d^2x}{dt^2}+b\dfrac{dx}{dt}+cx=0$

**Step 1:** Let the **trial solution** be $x=e^{mt}$

**Substitute** this back into $a\dfrac{d^2x}{dt^2}+b\dfrac{dx}{dt}+cx=0$

To get the **auxiliary equation** $am^2+bm+c=0$

**Step 2: Solve** the auxiliary equation to get $m_1$ and $m_2$

**Step 3: General solution** is $x=Ae^{m_1t}+Be^{m_2t}$

**Step 4:** The **Particular solution** is found by applying boundary conditions

## 2nd order homogeneous ODE $a\dfrac{d^2x}{dt^2}+b\dfrac{dx}{dt}+cx=0$

**Step 1**: Let the trial solution be $x=e^{mt}$ Now substitute this back into the ODE remembering that $\dfrac{dx}{dt}=me^{mt}=mx$ and $\dfrac{d^2x}{dt^2}=m^2e^{mt}=m^2x$

This is now called the auxiliary equation $am^2+bm+c=0$

**Step 2**: Solve the auxiliary equation for $m_1$ and $m_2$

**Step 3**: General solution is $x=Ae^{m_1t}+Be^{m_2t}$ or $x=e^{mt}(A+Bt)$ if $m_1=m_2$

For complex roots $m=\alpha\pm i\beta$ solution is $x=Ae^{(\alpha+i\beta)t}+Be^{(\alpha-i\beta)t}$ which is same as $x=e^{\alpha t}(Ae^{i\beta t}+Be^{-i\beta t})$ or $x=e^{\alpha t}(C\sin\beta t+D\cos\beta t)=Ee^{\alpha t}[\cos(\beta t+\phi)]$

**Step 4**: Particular solution is found now by applying boundary conditions

## Step 3 The General solution

**1** $x=Ae^{m_1t}+Be^{m_2t}$ If $m_1$ and $m_2$ are real

**2** $x=e^{mt}(A+Bt)$ If $m_1$ and $m_2$ are equal

**3** If $m_1$ and $m_2$ are complex $m=\alpha\pm i\beta$

$$x=Ae^{(\alpha+i\beta)t}+Be^{(\alpha-i\beta)t}$$

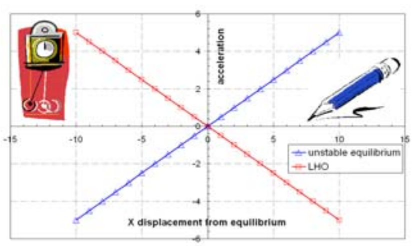
The solution simplifies $x=e^{\alpha t}(Ae^{i\beta t}+Be^{-i\beta t})$

$$x=e^{\alpha t}(C\sin\beta t+D\cos\beta t)=Ee^{\alpha t}[\cos(\beta t+\phi)]$$

## 2nd order homogeneous ODE

**Let's have a go now at the two most simple 2nd order homogeneous ODEs**

**Unstable equilibrium**




$$\frac{d^2}{dt^2}x(t)=\alpha^2x(t)$$

$$\frac{d^2}{dt^2}x(t)=-\omega_0^2x(t)$$

**Linear harmonic oscillator**

## 2nd order homogeneous ODE

**Example 3.2: Linear harmonic oscillator with boundary conditions that it has velocity V when time t=0 at x=0** $\dfrac{d^2x(t)}{dt^2}+\omega^2x(t)=0$

**Step 1:** Let the trial solution be $x=e^{mt}$ So $\dfrac{dx}{dt}=me^{mt}=mx$ and $\dfrac{d^2x}{dt^2}=m^2e^{mt}=m^2x$

**Step 2:** The auxiliary is then $m^2=-\omega^2$ and so roots are $m=\pm i\omega$

**Step 3**: General solution for complex $m=\alpha\pm i\beta$ is $x=e^{\alpha t}(C\sin\beta t+D\cos\beta t)$

where $\alpha=0$ and $\beta=\omega$ so $x=C\sin\omega t+D\cos\omega t$

**Step 4**: When x = 0 t = 0 so $0=C\sin\omega 0+D\cos\omega 0$ so $0=0+D$

From general solution we find $\dfrac{dx}{dt}=C\omega\cos\omega t-D\omega\sin\omega t$ Conditions state when t = 0 and x = 0, velocity is V, so $V=C\omega\cos\omega 0-D\omega\sin\omega 0=C\omega$ and so $C=\dfrac{V}{\omega}$

Particular solution is therefore $x=\dfrac{V}{\omega}\sin\omega t+0\cos\omega t=\dfrac{V}{\omega}\sin\omega t$

## 2nd order homogeneous ODE

**Example 3.3 Unstable equilibrium, with boundary conditions that it has velocity 0 at time t = 0 and x(0) = L** $\dfrac{d^2}{dt^2}x(t)-\alpha^2x(t)=0$

**Step 1:** Let the trial solution be $x=e^{mt}$ So $\dfrac{dx}{dt}=me^{mt}=mx$ and $\dfrac{d^2x}{dt^2}=m^2e^{mt}=m^2x$

**Step 2:** The auxiliary is then $m^2x=\alpha^2x$ and so roots are $m=\pm\alpha$

**Step 3**: General solution for real roots $m=\pm\alpha$ is $x(t)=Ae^{\alpha t}+Be^{-\alpha t}$

**Step 4**: When t = 0 x = L so $x(0)=Ae^{\alpha 0}+Be^{-\alpha 0}=L$ so $L=A+B$

From general solution we find $\dfrac{dx}{dt}=A\alpha e^{\alpha t}-B\alpha e^{-\alpha t}$ Conditions state when t = 0 and x = L, velocity is 0, so $\dfrac{dx}{dt}=0=A\alpha e^{\alpha 0}-B\alpha e^{-\alpha 0}=A\alpha-B\alpha$ so $B=A$

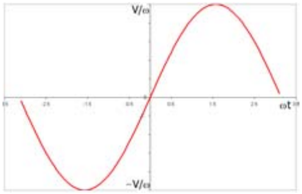
Particular solution is therefore $x(t)=\dfrac{L}{2}e^{\alpha t}+\dfrac{L}{2}e^{-\alpha t}=L\cosh(\alpha t)$

## 2nd order homogeneous ODE

**Example 3.2**
**Linear harmonic oscillator**

$$\frac{d^2}{dt^2}x(t) = -\omega_0{}^2 x(t)$$

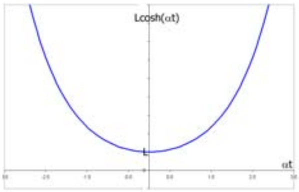
$$x(t) = \frac{V}{\omega}\sin\omega t$$




**Example 3.3**
**Unstable equilibrium**

$$\frac{d^2}{dt^2}x(t) = \alpha^2 x(t)$$

$$x(t) = \frac{L}{2}e^{\alpha t} + \frac{L}{2}e^{-\alpha t} = L\cosh(\alpha t)$$




## More examples

Find general solutions to $\dfrac{d^2y}{dx^2} = 4y$

$$\frac{d^2y}{dx} + 2\frac{dy}{dx} - 2y = 0$$

$$\frac{d^2y}{dx} - 2\frac{dy}{dx} + 2y = 0$$

And a particular solution to $3\dfrac{d^2y}{dx} - 2\dfrac{dy}{dx} + y = 0$

where $y(0) = 6$ and $\left.\dfrac{dy}{dx}\right|_{x=0} = 0$

## More examples 1 ans

Find general solutions to $\dfrac{d^2y}{dx^2} = 4y$

$$\frac{d^2y}{dx^2} = 4y$$

$$\frac{d^2y}{dx^2} - 4y = 0$$

Aux eqn: $M^2 - 4 = 0$

$M = \pm 2$

General Soln: $y = Ae^{2x} + Be^{-2x}$

## More examples 2 ans

Find general solutions to $\dfrac{d^2y}{dx} + 2\dfrac{dy}{dx} - 2y = 0$

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 2y = 0$$

Aux eqn: $M^2 + 2M - 2 = 0$

$M = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$M = \dfrac{-2 \pm \sqrt{4 + 8}}{2}$

$M = -1 \pm \sqrt{3}$

General Soln: $y = Ae^{(-1+\sqrt{3})x} + Be^{(-1-\sqrt{3})x}$

## More examples 3 ans

Find general solutions to $\dfrac{d^2y}{dx} - 2\dfrac{dy}{dx} + 2y = 0$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

Aux eqn: $M^2 - 2M + 2 = 0$

$M = \dfrac{2 \pm \sqrt{4 - 8}}{2}$

$M = \dfrac{2 \pm 2i}{2} = 1 \pm i$

General Soln: $y = De^{x}\cos(x + \phi)$

## More examples 4 ans

Find a particular solution to

$$3\frac{d^2y}{dx} - 2\frac{dy}{dx} + y = 0$$

where $y(0) = 6$

and $\left.\dfrac{dy}{dx}\right|_{x=0} = 0$

$$3\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Aux $3M^2 - 2M + 1 = 0$

$M = \dfrac{2 \pm \sqrt{4 - 12}}{6} = \frac{1}{3}(1 \pm i\sqrt{2})$

$y = e^{x/3}(D\cos(\sqrt{2}x + \phi))$

$x = 0 \quad y = 6$

$6 = D\cos\phi$

$\dfrac{dy}{dx} = \frac{1}{3}e^{x/3}(-D\sqrt{2}\sin(\sqrt{2}x + \phi)) +$

$0 = -\frac{1}{3}D\sqrt{2}\sin\phi \Rightarrow \phi = 0$

$D = 6$

$y = 6e^{x/3}\cos(\sqrt{2}x)$

## Lecture 3 summary ….

For equations like this $a\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0$

Use a trial solution $x = e^{mt}$

Solve the auxiliary equation and write down the general solution $x = Ae^{m_1 t} + Be^{m_2 t}$

# Lecture 4
# Ordinary Differential Equations

Purpose of lecture 4:

- Solve the full 2nd order homogeneous ODE
- With reference to damped harmonic oscillators
- Introduce method for solving in-homogeneous ODE

## 2nd order homogeneous ODE - revision

**Solving** $a\dfrac{d^2x}{dt^2} + b\dfrac{dx}{dt} + cx = 0$

**Step 1**: Let the trial solution be $x = e^{mt}$ Now substitute this back into the ODE remembering that $\dfrac{dx}{dt} = me^{mt} = mx$ and $\dfrac{d^2x}{dt^2} = m^2e^{mt} = m^2x$

This is now called the auxiliary equation $am^2 + bm + c = 0$

**Step 2**: Solve the auxiliary equation for $m_1$ and $m_2$

**Step 3**: General solution is $x = Ae^{m_1 t} + Be^{m_2 t}$ or $x = e^{mt}(A + Bt)$ if $m_1 = m_2$

For complex roots $m = \alpha \pm i\beta$ solution is $x = Ae^{(\alpha+i\beta)t} + Be^{(\alpha-i\beta)t}$ which is same as $x = e^{\alpha t}(Ae^{i\beta t} + Be^{-i\beta t})$ or $x = e^{\alpha t}(C\sin\beta t + D\cos\beta t) = Ee^{\alpha t}[\cos(\beta t + \phi)]$

**Step 4**: Particular solution is found now by applying boundary conditions

## 2nd order homogeneous ODE

**Solve** $\dfrac{d^2x}{dt^2} - 4\dfrac{dx}{dt} + 4x = 0$ **Boundary conditions x=4, velocity=0 when t=0**

**Step 1**: Trial solution is $x = e^{mt}$ so auxiliary equation $m^2 - 4m + 4 = 0$

**Step 2**: Solving the auxiliary equation gives $m_1 = m_2 = 2$

**Step 3**: General solution is $x = e^{mt}(A + Bt)$ if $m_1 = m_2$ giving $x = e^{2t}(A + Bt)$

**Step 4**: Particular solution is found now by applying boundary conditions

When t = 0, x = 4 so $4 = e^0(A + B0)$ and so $4 = A$

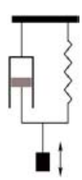
Since $x = e^{2t}(A + Bt) = Ae^{2t} + Bte^{2t}$ velocity = $\dfrac{dx}{dt} = 2Ae^{2t} + Be^{2t} + 2Bte^{2t}$

Boundary conditions give $0 = 2Ae^0 + Be^0 + 2B0e^0 = 2A + B$ so $B = -2A = -8$

Full solution is therefore $x = e^{2t}(4 - 8t)$

## 2nd order homogeneous ODE

### Example 3: Damped harmonic oscillators



Effect of damper drag will be a function of $\dfrac{dx}{dt}$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0$$

**Rally suspension**
http://uk.youtube.com/watch?v=CZeCuS4xzL0&feature=related

**Modelling differential equations**
http://www.falstad.com/diffeq/

**Crosswind landing**
http://uk.youtube.com/watch?v=n9yF09DMrrl

## 2nd order homogeneous ODE

**Example 3: Linear harmonic oscillator with damping** $\dfrac{d^2x}{dt^2} + 2\gamma\dfrac{dx}{dt} + \omega_0^2 x = 0$

**Step 1:** Let the trial solution be $x = e^{mt}$ So $\dfrac{dx}{dt} = me^{mt} = mx$ and $\dfrac{d^2x}{dt^2} = m^2e^{mt} = m^2x$

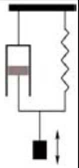
**Step 2:** The auxiliary is then $m^2 + 2\gamma m + \omega_0^2 = 0$ with roots $m = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}$

**Step 3**: General solution is then……. HANG ON!!!!!

**If $\gamma^2 > \omega_0^2$ then $\sqrt{\gamma^2 - \omega_0^2}$ will always be real**

**What if $\gamma^2 < \omega_0^2$ or $\gamma^2 = \omega_0^2$**

## 2nd order homogeneous ODE



Example 3: Damped harmonic oscillator $\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=0$

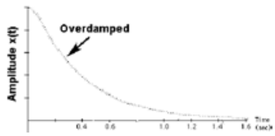
Auxiliary is $m^2+2\gamma m+\omega_0^2=0$ roots are $m=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}$

**BE CAREFUL – THERE ARE THREE DIFFERENT CASES!!!!!**

**(i)** Over-damped $\gamma^2>\omega_0^2$ gives two real roots $m_1=-\gamma+\sqrt{\gamma^2-\omega_0^2}$

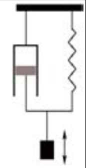
$m_2=-\gamma-\sqrt{\gamma^2-\omega_0^2}$

$$x=Ae^{m_1t}+Be^{m_2t}$$




Both $m_1$ and $m_2$ are negative so $x(t)$ is the sum of two exponential decay terms and so tends pretty quickly, to zero. The effect of the spring has been made of secondary importance to the huge damping, e.g. aircraft suspension

## 2nd order homogeneous ODE



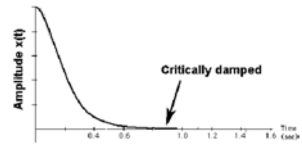
Example 3: Damped harmonic oscillator $\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=0$

Auxiliary is $m^2+2\gamma m+\omega_0^2=0$ roots are $m=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}$

**BE CAREFUL – THERE ARE THREE DIFFERENT CASES!!!!!**

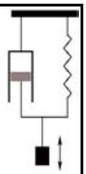
**(ii)** Critically damped $\gamma^2=\omega_0^2$ gives a single root $m_1=m_2=-\gamma$

$$x=e^{-\gamma t}(A+Bt)$$




Here the damping has been reduced a little so the spring can act to change the displacement quicker. However the damping is still high enough that the displacement does not pass through the equilibrium position, e.g. car suspension.

## 2nd order homogeneous ODE



Example 3: Damped harmonic oscillator $\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=0$

Auxiliary is $m^2+2\gamma m+\omega_0^2=0$ roots are $m=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}$

**BE CAREFUL – THERE ARE THREE DIFFERENT CASES!!!!!**

**(iii)** Under-damped $\gamma^2<\omega_0^2$ This will yield complex solutions due to presence of square root of a negative number.

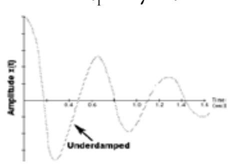
Let $\Omega^2=\omega_0^2-\gamma^2$ so $\sqrt{\gamma^2-\omega_0^2}=\sqrt{-\Omega^2}=\pm i\Omega$ thus $m_2=-\gamma-i\Omega$

<u>We do this so that Ω is real</u>

$m_1=-\gamma+i\Omega$

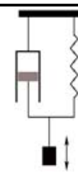
As before general solution with complex roots can be written as

$$x=Ee^{-\gamma t}[\cos(\Omega t+\phi)]$$




The solution is the product of a sinusoidal term and an exponential decay term – so represents sinusoidal oscillations of decreasing amplitude. E.g. bed springs.

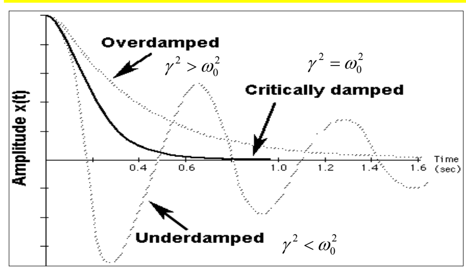
## 2nd order homogeneous ODE



Example 3: Damped harmonic oscillator $\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=0$

Auxiliary is $m^2+2\gamma m+\omega_0^2=0$ roots are $m=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}$

**BE CAREFUL – THERE ARE THREE DIFFERENT CASES!!!!!**




Q factor

$$Q=\frac{\omega_0}{2\gamma}$$

**A high Q factor means the oscillations will die more slowly**

# 2nd order inhomogenous ODE

Introduction

Hopefully these equations from PHY102 Waves & Quanta are familiar to you….

### Forced oscillation without damping:

$$m\frac{d^2}{dt^2}x(t)+kx(t)=H_0\cos\omega_Dt$$

### Forced Oscillation with damping:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=H_0\cos\omega_Dt$$

e.g. Mechanical oscillators, LCR circuits, optics and lasers, nuclear physics, …

## Inhomogeneous ordinary differential equations

$$p\frac{d^2x}{dt^2}+q\frac{dx}{dt}+rx=f(t)$$

<u>Step 1</u>: Find the general solution to the related homogeneous equation and call it the complementary solution $x_c(t)$

<u>Step 2</u>: Find the particular solution $x_p(t)$ of the equation by substituting an appropriate trial solution into the <u>full</u> original inhomogeneous ODE.

| | | | | |
|---|---|---|---|---|
| e.g. | If | $f(t) = t^2$ | try | $x_p(t) = at^2 + bt + c$ |
| | If | $f(t) = 5e^{3t}$ | try | $x_p(t) = ae^{3t}$ |
| | If | $f(t) = 5e^{i\omega t}$ | try | $x_p(t) = ae^{i\omega t}$ |
| | If | $f(t) = \sin 2t$ | try | $x_p(t) = a(\cos 2t) + b(\sin 2t)$ |
| | If | $f(t) = \cos \omega t$ | try | $x_p(t) = \mathrm{Re}[ae^{i\omega t}]$ see later for explanation |
| | If | $f(t) = \sin \omega t$ | try | $x_p(t) = \mathrm{Im}[ae^{i\omega t}]$ |

If your trial solution has the correct form, substituting it into the differential equation will yield the values of the constants *a, b, c*, etc.

<u>Step 3</u>: The complete general solution is then $x(t)=x_c(t)+x_p(t)$ .

<u>Step 4</u>: Apply boundary conditions to find the values of the constants

$$p\frac{d^2x}{dt^2}+q\frac{dx}{dt}+rx=f(t)$$

**What is the form of the particular solution for the following functions:**

$$f(t)=3t^2$$

$$f(t)=3\sin 4t$$

$$f(t)=3e^{4it}$$

$$f(t)=F\cos\omega t$$

## Particular solutions:

$$f(t)=3t^2 \quad x_c(t)=at^2+bt+c$$

$$f(t)=3\sin 4t \quad x_c(t)=a\sin 4t+b\cos 4t$$

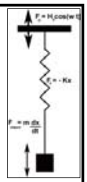
$$f(t)=3e^{4it} \quad x_c(t)=ae^{4it}$$

$$f(t)=F\cos\omega t \quad x_c(t)=a\cos\omega t+b\sin\omega t$$

## Inhomogeneous ODEs

**Example 4: Undamped driven oscillator**
**At rest when t = 0 and x = L**

$$\frac{d^2}{dt^2}x(t)+\omega_0^2x(t)=F\cos\omega t$$



**Step 1: The corresponding homogeneous general equation is the LHO that we solved in last lecture, therefore** $x_c(t)=A\cos\omega_0 t+B\sin\omega_0 t$

**Step 2: For** $F\cos\omega t$ **we should use trial solution** $x_p(t)=a\cos\omega t+b\sin\omega t$

**Putting this into FULL ODE gives:-**

$$(\omega_0^2-\omega^2)a\cos\omega t+(\omega_0^2-\omega^2)b\sin\omega t=F\cos\omega t$$

**Comparing terms we can say that b = 0 and**

$$a=\frac{F}{(\omega_0^2-\omega^2)}$$

**So** $x_p(t)=\dfrac{F}{\omega_0^2-\omega^2}\cos\omega t$

**Step 3: So full solution is** $x(t)=A\cos\omega_0 t+B\sin\omega_0 t+\dfrac{F}{\omega_0^2-\omega^2}\cos\omega t$

### Inhomogeneous ordinary differential equations

**Example 4: Undamped driven oscillator**
**At rest when t = 0 and x = L**

$$\frac{d^2}{dt^2}x(t)+\omega_0^2x(t)=F\cos\omega t$$

**Step 4: Apply boundary conditions to find the values of the constants A and B for the full solution** $x(t)=A\cos\omega_0 t+B\sin\omega_0 t+\dfrac{F}{\omega_0^2-\omega^2}\cos\omega t$

**When t = 0, x = L so** $L(0)=A\cos\omega_0 0+B\sin\omega_0 0+\dfrac{F}{\omega_0^2-\omega^2}\cos\omega 0=A+\dfrac{F}{\omega_0^2-\omega^2}$

$$A=L-\frac{F}{\omega_0^2-\omega^2}$$

### Inhomogeneous ordinary differential equations

**Example 4: Undamped driven oscillator**
**At rest when t = 0 and x = L**

$$\frac{d^2}{dt^2}x(t)+\omega_0^2x(t)=F\cos\omega t$$

**Step 4: Apply boundary conditions to find the values of the constant B**

**At rest means velocity** $\dfrac{dx}{dt}=0$ **at x = L and t = 0. So differentiate full solution**

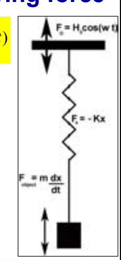
$$\frac{dx(t)}{dt}=-A\omega_0\sin\omega_0 t+B\omega_0\cos\omega_0 t-\frac{F\omega}{\omega_0^2-\omega^2}\sin\omega t$$ **and as velocity = 0 when t = 0**

$$\frac{dx(t)}{dt}=-A\omega_0\sin\omega_0 0+B\omega_0\cos\omega_0 0-\frac{F\omega}{\omega_0^2-\omega^2}\sin\omega 0=B\omega_0=0$$ **so** $B=0$

**Full solution is therefore** $x(t)=\left[L-\dfrac{F}{\omega_0^2-\omega^2}\right]\cos\omega_0 t+\dfrac{F}{\omega_0^2-\omega^2}\cos\omega t$

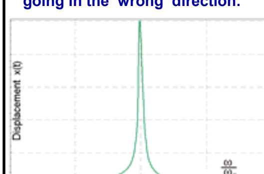
## Undamped driven oscillator with SHM driving force

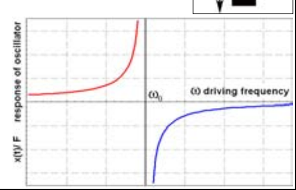
**This can also be written as** $x(t)=L\cos\omega_0 t+\dfrac{F}{(\omega_0^2-\omega^2)}(\cos\omega t-\cos\omega_0 t)$



**A few comments**

- **Note that the solution is clearly not valid for** $\omega=\omega_0$
- **The ratio** $\dfrac{x(t)}{F}$ **is sometimes called the response of the oscillator. It is a function of** $\omega$**. It is positive for** $\omega<\omega_0$**, negative for** $\omega>\omega_0$**. This means that at low frequency the oscillator follows the driving force but at high frequencies it is always going in the 'wrong' direction.**



## Lecture 5 – Inhomogeneous ODE

$$p\frac{d^2x}{dt^2}+q\frac{dx}{dt}+rx=f(t)$$

## Inhomogeneous ordinary differential equations

$$p\frac{d^2x}{dt^2}+q\frac{dx}{dt}+rx=f(t)$$

**Step 1**: Find the general solution to the related homogeneous equation and call it the complementary solution $x_c(t)$

**Step 2**: Find the particular solution $x_p(t)$ of the equation by substituting an appropriate trial solution into the full original inhomogeneous ODE.

| e.g. | | | |
|---|---|---|---|
| If | $f(t) = t^2$ | try | $x_p(t) = at^2 + bt + c$ |
| If | $f(t) = 5e^{3t}$ | try | $x_p(t) = ae^{3t}$ |
| If | $f(t) = 5e^{i\omega t}$ | try | $x_p(t) = ae^{i\omega t}$ |
| If | $f(t) = \sin 2t$ | try | $x_p(t) = a(\cos 2t) + b(\sin 2t)$ |
| If | $f(t) = \cos \omega t$ | try | $x_p(t) = \mathrm{Re}[ae^{i\omega t}]$ see later for explanation |
| If | $f(t) = \sin \omega t$ | try | $x_p(t) = \mathrm{Im}[ae^{i\omega t}]$ |

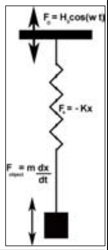
If your trial solution has the correct form, substituting it into the differential equation will yield the values of the constants *a, b, c*, etc.

**Step 3**: The complete general solution is then $x(t)=x_c(t)+x_p(t)$ .

**Step 4**: Apply boundary conditions to find the values of the constants

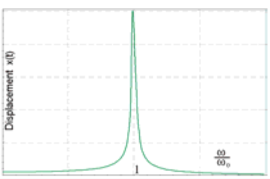
## Undamped driven oscillator with SHM driving force

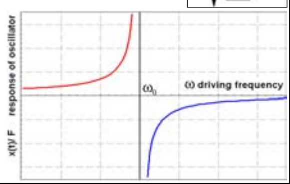
This can also be written as $x(t)=L\cos\omega_0 t+\frac{F}{(\omega_0^2-\omega^2)}(\cos\omega t-\cos\omega_0 t)$




A few comments

- Note that the solution is clearly not valid for $\omega=\omega_0$
- The ratio $\frac{x(t)}{F}$ is sometimes called the response of the oscillator. It is a function of $\omega$. It is positive for $\omega < \omega_0$, negative for $\omega > \omega_0$ . This means that at low frequency the oscillator follows the driving force but at high frequencies it is always going in the 'wrong' direction.

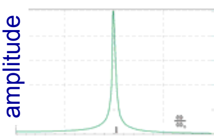


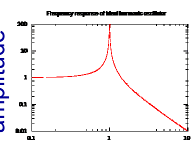

## Undamped driven oscillator with SHM driving force

$$x(t)=L\cos\omega_0 t+\frac{F}{(\omega_0^2-\omega^2)}(\cos\omega t-\cos\omega_0 t)$$

amplitude



amplitude



$$amplitude=\left|\frac{1}{1-\left(\frac{\omega^2}{\omega_0^2}\right)}\right|$$

$$response=\frac{1}{1-\left(\frac{\omega^2}{\omega_0^2}\right)}$$

## Example of inhomogeneous ODE

Solve $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=2\sin(4t)$

***FIRST*** Find complementary solution of related homogeneous equation $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=0$

**Step 1: With trial solution** $x=e^{mt}$ **auxiliary is** $m^2+4m+3=0$

**Step 2: Soln of auxiliary to complementary homo ODE** $m_1=-3$, $m_2=-1$

**Step 3: Complementary solution is** $x_c(t)=Ae^{-3t}+Be^{-t}$

## Find the particular solution $\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=2\sin(4t)$

**Step 4: Use the trial solution** $x_p(t)=a\cos 4t+b\sin 4t$ **and substitute it in the FULL differential equation**

$$\frac{dx}{dt}=-4a\sin 4t+4b\cos 4t \qquad \frac{d^2x}{dt^2}=-16a\cos 4t-16b\sin 4t$$

**so**

$$(-16a\cos 4t-16b\sin 4t)+4(-4a\sin 4t+4b\cos 4t)+3(a\cos 4t+b\sin 4t)=2\sin 4t$$

**Collecting sine and cosine terms**

$$16b\cos 4t-13a\cos 4t-13b\sin 4t-16a\sin 4t=2\sin 4t$$

$$(16b-13a)\cos 4t+(-13b-16a)\sin 4t=2\sin 4t$$

## Example of inhomogeneous ODE

$$\frac{d^2x}{dt^2}+4\frac{dx}{dt}+3x=2\sin(4t)$$

**Having collected terms …**

$$(16b-13a)\cos 4t+(-13b-16a)\sin 4t=2\sin 4t$$

**We compare sides to give …**

$$(16b-13a)=0 \quad and \quad (-13b-16a)=2$$

**Solving for *a* and *b*** $a=-\dfrac{32}{425} \quad and \quad b=-\dfrac{26}{425}$

**Step 5: General solution is**

$$x(t)=x_c(t)+x_p(t)=Ae^{-3t}+Be^{-t}-\frac{32}{425}\cos 4t-\frac{26}{425}\sin 4t$$

# Driven damped harmonic motion

**4x4 with no shock absorbers**

http://uk.youtube.com/watch?v=AKcTA5j6K80&feature=related

**Bose suspension**

http://uk.youtube.com/watch?v=eSi6J-QK1lw

## Finding a partial solution to inhomogeneous ODE using complex form

Sometimes it's easier to use complex numbers rather than messy algebra

**We can write** $Fe^{i\omega t}=F\cos\omega t+iF\sin\omega t$

**and we can also say** $\mathrm{Re}\{Fe^{i\omega t}\}=F\cos\omega t$

**and** $\mathrm{Im}\{Fe^{i\omega t}\}=F\sin\omega t$

## Finding a partial solution to inhomogeneous ODE using complex form

Let's look again at example 4 of lecture 4 notes $\dfrac{d^2}{dt^2}x(t)+\omega_0^2x(t)=F\cos\omega t$

But let's choose to solve the equation $\dfrac{d^2}{dt^2}X(t)+\omega_0^2X(t)=Fe^{i\omega t}$

If we solve for $X(t)$ and take only the real coefficient then this will be a solution for $x(t)$ *since* $\mathrm{Re}\{Fe^{i\omega t}\}=F\cos\omega t$

## Finding a particular solution to inhomogeneous ODE using complex form

$$\frac{d^2}{dt^2}x(t)+\omega_0^2x(t)=F\cos\omega t$$

$$\frac{d^2}{dt^2}X(t)+\omega_0^2X(t)=Fe^{i\omega t}$$

*Trial particular solution*

$X_p(t)=Ae^{i\omega t}$ $\quad \dfrac{dX_p}{dt}=Ai\omega e^{i\omega t}$ $\quad \dfrac{d^2X_p}{dt^2}=-A\omega^2e^{i\omega t}$

Substituting gives $(\dfrac{d^2}{dt^2}+\omega_0^2)Ae^{i\omega t}=(-\omega^2+\omega_0^2)Ae^{i\omega t}=Fe^{i\omega t}$

So $A=\dfrac{F}{(\omega_0^2-\omega^2)}$

Therefore since $X_p(t)=\dfrac{F}{(\omega_0^2-\omega^2)}e^{i\omega t}$ and $x_p(t)=\mathrm{Re}\{X_p(t)\}=\dfrac{F\cos\omega t}{(\omega_0^2-\omega^2)}$

## Example

$$\frac{d^2x(t)}{dt^2}-4x(t)=5\sin 8t$$

1. Determine the **complementary solution**
2. Select a trail form of a **particular solution**
3. Substitute the trial in the ODE to determine the unknown coefficients in the **particular solution**
4. Write down the **complete general solution**

Left side of class : use *sine + cosine*, right side of class: use *complex* method

## Find full solution of inhomoODE extra example below

$$\frac{d^2x(t)}{dt^2} - 4x(t) = 5\sin 8t$$

**Step1 and 2: For trial of** $x = e^{mt}$ **auxiliary is** $m^2 - 4 = 0$ **roots** $m_1 = 2, m_2 = -2$

**Step 3: Complementary solution is** $x_c(t) = Ae^{2t} + Be^{-2t}$

**Step 4: Use the trial solution** $x_p(t) = a\cos 8t + b\sin 8t$ **and substitute it in FULL expression.** $\frac{dx}{dt} = -8a\sin 8t + 8b\cos 8t$ $\frac{d^2x}{dt^2} = -64a\cos 8t - 64b\sin 8t$

**so** $(-64a\cos 8t - 64b\sin 8t) - 4(a\cos 8t + b\sin 8t) = 5\sin 8t$

**cancelling** $-68a\cos 8t - 68b\sin 8t = 5\sin 8t$

**Comparing sides gives….** $-68b = 5 \quad and \quad -68a = 0$

**Solving gives** $a = 0 \quad and \quad b = -\frac{5}{68}$ **so** $x_p(t) = -\frac{5}{68}\sin 8t$

**Step 5: General solution is** $x(t) = x_c(t) + x_p(t) = Ae^{2t} + Be^{-2t} - \frac{5}{68}\sin 8t$

## Find partial solution to example of inhomoODE using complex form

Complementary solution is unchanged so . . . . . $\frac{d^2x(t)}{dt^2} - 4x(t) = 5\sin 8t$ **Is it faster this way?**

**Step 4**: Imagine equation we want to solve is of the form $\frac{d^2X}{dt^2} - 4X = 5e^{i8t}$

$\mathrm{Im}\{5e^{i8t}\} = 5\sin 8t \quad 5e^{i8t} = 5\cos 8t + 5i\sin 8t$

So we pick a trial solution $X_p(t) = Ae^{i8t}$ $\frac{dX}{dt} = i8A\omega e^{i8t}$ $\frac{d^2X}{dt^2} = -64Ae^{i8t}$

Substituting into FULL equation gives $-64Ae^{i8t} - 4Ae^{i8t} = -68Ae^{i8t} = 5e^{i8t}$

Cancelling gives $A = -\frac{5}{68}$ and $X_p(t) = -\frac{5}{68}e^{8it}$

**Solution is** $x_p(t) = \mathrm{Im}\{X_p(t)\} = \mathrm{Im}\left\{-\frac{5}{68}e^{8it}\right\} = \mathrm{Im}\left\{-\frac{5}{68}(\cos 8t + i\sin 8t)\right\} = -\frac{5}{68}\sin 8t$

## ODE – summary and tips

- Learn the steps and follow them
- Write neatly
- Don’t work too many algebraic steps at once
- Practice many questions