Topic 2

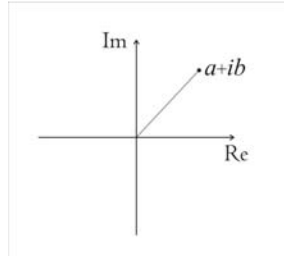
Complex numbers

Purpose of lecture

- **Remind you of Cartesian and polar forms**
- **Conversion between them**
- **Powers and roots of complex numbers**
- **Integration by parts**

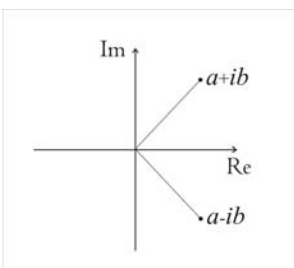
The Argand diagram

$$z = a + ib$$



Complex conjugate

$$z^* = a - ib$$

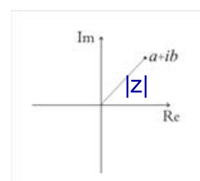


Why complex conjugate?

If $z = a + ib$ Evaluate z^2

zz^*

How do we find $|z|$?



Why complex conjugate?

What is |z|?
What works - z^2 or zz^*?

$$z^2 = (a+ib)^2 = a^2 - b^2 + 2iab$$

$$zz^* = (a+ib)(a-ib) = a^2 + b^2$$

Real answers from imaginary numbers

$$|z| = \sqrt{zz^*}$$

$$|z| = \sqrt{a^2 + b^2}$$

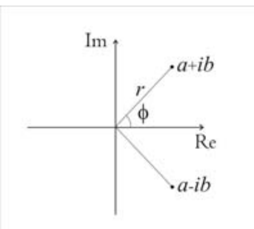
Example 2.1

$$|3+4i|$$

Example 2.1 (ans)

$$|3+4i| = \sqrt{(3+4i)(3-4i)}$$
$$|3+4i| = \sqrt{9+16} = 5$$

Polar form



$$a = r\cos\phi$$
$$b = r\sin\phi$$
$$\phi = \tan^{-1}\left(\frac{b}{a}\right)$$
$$r = \sqrt{a^2+b^2}$$

Why both forms?

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$$
$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2)$$
$$z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

Why both forms?

Multiplication?

Division?

Why both forms?

$$z_1 z_2 = r_1 r_2 e^{i(\phi_1+\phi_2)}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\phi_1-\phi_2)}$$

Example 2.2

$$\frac{1+i}{1+1.73i}$$

$$\tan^{-1}(1.73) = \pi/3$$

$$\tan^{-1}(1) = \pi/4$$

Example 2.2 (ans)

$z_1 = 1+i = \sqrt{2}e^{i\pi/4}$

$\tan\theta = \frac{1}{1}$ $\quad r = \sqrt{2}$

$\theta = \pi/4$

$z_2 = 1+1.73i$

$\tan\theta = 1.73$ $\quad r = \sqrt{1+3} = 2$

$\theta = \pi/3$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{i\pi/3}} = \frac{\sqrt{2}}{2}e^{i(\pi/4-\pi/3)}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2}e^{-i\pi/12} = 0.707e^{-i\pi/12}$$

Powers and roots

$$z^n = r^n e^{in\phi}$$

$$z^{1/n} = r^{1/n}e^{i(\phi+2\pi p)/n}$$

Multiple roots

$$\sqrt{1}$$

$$\sqrt[3]{1}$$

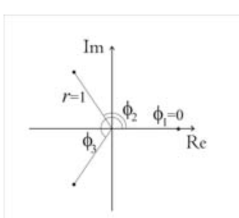
$$\sqrt[n]{1}$$

Multiple roots

$$\sqrt[3]{1} = \begin{cases} 1 \\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i \\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i \end{cases}$$

$$z = e^{\frac{2i\pi p}{3}}$$

$$z = 1, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}$$



Multiple roots

$$\sqrt[n]{1} = 1^{1/n}$$

$$z^{1/n}$$

n roots

Roots are easy $z^{1/n} = r^{1/n} e^{i(\phi + 2\pi p)/n}$

Step 1: how many roots will there be?

Step 2: how many values of p?

Step 3: work them out one at a time…

Example 2.3 (Q)

If $z = 9e^{i\pi/3}$ find $z^{1/2}$

Example 2.3 (A)

Step 1: write down z in polars with the $2\pi p$ bit added on to the argument. $z = 9e^{i(\pi/3 + 2\pi p)}$

Step 2: say how many values of p you'll need and write out the rooted expression here n = 2 so I'll need 2 values of p; p = 0 and p = 1 $z^{½} = \sqrt{9}\, e^{i(\pi/3 + 2\pi p)/2}$

Step 3: Work it out for each value of p: $z^{½} = 3e^{i(\pi/3)/2} = 3e^{i(\pi/6)}$ for p = 0, $z^{½} = 3e^{i(\pi/3 + 2\pi)/2} = 3e^{i(\pi/6 + \pi)}$ for p = 1

Remember that $e^{i\phi} = (\cos\phi + i\sin\phi)$ so $e^{i\pi} = -1$

Therefore we could write
$z^{½} = 3e^{i(\pi/6 + \pi)} = 3e^{i\pi/6}(e^{i\pi}) = -3e^{i\pi/6}$ for p =1, and $3e^{i(\pi/6)}$ for p = 0

Example 2.4 (Q)

If $z = 27^{i\pi/2}$ find $z^{1/3}$

Example 2.4 (A)

Step 1 : $z = 27e^{i(\pi/2 + 2\pi p)}$

Step 2 : for n=3 p=0, 1, 2

Step 3 : for p=0 $z^{1/3} = 3e^{i(\pi/2)/3} = 3e^{i\pi/6}$

for p=1 $z^{1/3} = 3e^{i(\pi/2 + 2\pi)/3} = 3e^{i(5\pi/6)}$

for p=2 $z^{1/3} = 3e^{i(\pi/2 + 4\pi)/3} = 3e^{i(9\pi/6)}$

So the 3 roots of $z^{1/3}$ are : $3e^{i\pi/6}, 3e^{i5\pi/6}, 3e^{i9\pi/6}$

Complex trig functions

Remember

$$e^{ikx} = \cos kx + i\sin kx$$

$$e^{-ikx} = \cos kx - i\sin kx$$

Complex trig functions

Remember

$$e^{ikx} = \cos kx + i\sin kx$$

$$e^{-ikx} = \cos kx - i\sin kx$$

$$\cos kx = \tfrac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$$

$$\sin kx = \tfrac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$$

Complex differentiation

Remember that i is just a number!

$$\frac{d(e^{ikx})}{dx} = ike^{ikx}$$

Much easier than sine's and cosine's

Integration by parts - Remember the chain rule

$$\frac{d(f(x)g(x))}{dx} = f(x)\frac{d(g(x))}{dx} + \frac{d(f(x))}{dx}g(x)$$

$$f(x)g(x) = \int f(x)\frac{d(g(x))}{dx}dx + \int \frac{d(f(x))}{dx}g(x)dx$$

$$\int f(x)\frac{d(g(x))}{dx}dx = f(x)g(x) - \int g(x)\frac{d(f(x))}{dx}dx$$

$$v = g(x) \qquad u = f(x)$$

$$\frac{dv}{dx} = \frac{d(g(x))}{dx} \qquad \frac{du}{dx} = \frac{d(f(x))}{dx}$$

$$v = \int \frac{d(g(x))}{dx}dx \qquad du = \frac{d(f(x))}{dx}dx$$

$$\int udv = uv - \int vdu$$

Integration by parts - example

$$\int udv = uv - \int vdu$$

$$\int x\cos xdx$$

$$u = x \qquad dv = \cos xdx$$

$$\frac{du}{dx} = 1 \qquad v = \int \cos xdx$$

$$du = dx \qquad v = \sin x$$

$$\int x\cos xdx = x\sin x - \int \sin xdx$$

$$\int x\cos xdx = x\sin x + \cos x + c$$