PHY226 Tutorial Questions

Week 2: Binomial Series and Complex numbers

1. (a) Find the first few terms in the binomial series expansion of $(1-x)^{-1/2}$.

   (b) In the theory of special relativity, an object moving with velocity v has mass $m = \dfrac{m_o}{\sqrt{1-v^2/c^2}}$ where m_0 is the mass of the object at rest and c is the speed of light. The kinetic energy of the object is the difference between its total energy and its energy at rest: $K = mc^2 - m_0c^2$. Using the series found in (a), show that when $v << c$, the relativistic expression for K agrees with the classical expression $K = ½\, mv^2$.

① a) Expand $(1-x)^{-\frac{1}{2}}$

Binomial $(1+x)^n = 1 + nx + \dfrac{n(n-1)x^2}{2!} + \dfrac{n(n-1)(n-2)x^3}{3!}$

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\tfrac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-x)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-x)^3}{3!}$$

$$= 1 + \frac{x}{2} + \frac{3}{8}x^2 + \frac{5}{16}x^3$$

b) if $v \ll c$ then the series applies well

Take first two terms

$(1-x)^{-\frac{1}{2}} = 1 + \dfrac{x}{2}$ Since $M = M_0\left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}}$

$x = \dfrac{v^2}{c^2}$

$M = M_0\left(1 + \dfrac{v^2}{2c^2}\right)$

$k = Mc^2 - M_0c^2 = \frac{1}{2}M_0v^2$

(c) What is $z^{\frac{1}{3}}$ if $z = 64e^{\frac{i\pi}{3}}$?

① c) $z = 64e^{i\pi/3}$ find $z^{1/3}$ Three roots reqd

$z^{1/3} = ((64)^{1/3} e^{i(\frac{\pi}{3}+2\pi p)/3}$ where $p = 0, 1, 2$

$z^{1/3} = 4e^{i\pi/9}, \; 4e^{7i\pi/9}, \; 4e^{13i\pi/9}$

Week 3: First and second order ODEs

2. Make sure you can state (hopefully instantly!) the general solution of the following 1st order ODEs:
   (a) $\frac{dx}{dt} = -\alpha x$, (b) $\frac{dx}{dt} = \beta x$, (c) $\frac{dx}{dt} = i\gamma x$, where α, β, γ are real positive constants.
   In what physical situations might equations of forms (a) and (b) occur?

3. What is the solution of the equation $\frac{dN(t)}{dt} = 0$ subject to the condition that $N = N_0$ when $t = 0$?

4. 2nd order ODEs. In lecture 4 we discussed the equations and solutions for the linear harmonic oscillator and unstable equilibrium.
   Now consider the 1D time independent Schrodinger equation for the spatial wavefunction *u(x)* of a particle of energy E in a constant potential V: $-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + Vu = Eu$.
   Which form does this equation have, and what do its solutions look like, (i) for V < E? (ii) for V > E?

② a) $\frac{dx}{dt} = -\alpha x \qquad x = Ae^{-\alpha t}$ where A is a constant

b) $\frac{dx}{dt} = \beta x \qquad x = Ae^{\beta t}$ "

c) $\frac{dx}{dt} = i\gamma x \qquad x = Ae^{i\gamma t}$ "

a – Radioactive decay
b – Bacterial growth

③ $\frac{dN(t)}{dt} = 0$ where $N = N_0$ at $t = 0$

$N = N_0$ always

④ $-\frac{\hbar^2}{2m}\frac{d^2u}{dx^2} + Vu = Eu$ is a 2nd order homogenous ODE of form $a\frac{d^2u}{dx^2} + bu = 0$

$a = -\frac{\hbar^2}{2m}$, $b = V - E$

If $V < E$ then b is -ve
then Sol^n is sum of exponentials and unstable $u(x) = Ae^{\alpha x} + Be^{-\alpha x}$

If $V > E$ then b is +ve
Sol^n is wavelike and stable $u(x) = A\cos\omega t + B\sin\omega t$

Week 4: ODEs cont.

5. (a) Express the complex function $Z(\omega) = -\omega^2 + 2i\gamma\omega + \omega_0^2$ in the form $Z(\omega) = |Z(\omega)|e^{i\phi}$ by finding expressions for $|Z(\omega)|$ and $\tan\phi$.

⑤ a) $Z(\omega) = -\omega^2 + 2i\gamma\omega + \omega_0^2$

$Z(\omega) = (\omega_0^2 - \omega^2) + i(2\gamma\omega)$ of form $a+ib$

$Z = |z|e^{i\phi}$

$|z| = \sqrt{a^2+b^2}$

$\phi = \tan^{-1}\left(\frac{b}{a}\right)$

$|z| = \sqrt{(\omega_0^2-\omega^2)^2+4\gamma^2\omega^2}$

$$\boxed{\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)}$$

(b) In the lecture we said that if $X(t)$ is a solution of the equation $a\frac{d^2X}{dt^2}+b\frac{dX}{dt}+cX = Fe^{i\omega t}$ then $x(t) = \mathrm{Re}[X(t)]$ is the solution of $a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx = F\cos\omega t$. Can you justify this?

b) The solution to $a\frac{d^2X}{dt^2} + b\frac{dX}{dt} + cX = Fe^{i\omega t}$

has sol^n $X = X_c + X_p$

where X_c is the solution to $a\frac{d^2X}{dt^2} + b\frac{dX}{dt} + cX = 0$

and is always real (one of three forms)

X_p is the particular sol^n and has form $ge^{i\omega t}$

$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = F\cos\omega t$ has solution $x_c + x_p$

Since $x_c = X_c$ only the particular sol^n can contain imaginary terms and if the driving term is real the x_p will be real. $\mathrm{Re}(Fe^{i\omega t}) = \mathrm{Re}(F\cos\omega t + i\sin\omega t) = F\cos\omega t$

(c) Using complex exponentials, find the steady state solution of the damped, driven oscillator equation $\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = F\cos\omega t$.

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = F\cos\omega t = Re\left(Fe^{i\omega t}\right)$$

Steady State Sol^n is particular solution x_p.

Using Complex form $\quad x_p = Re\left(g e^{i\omega t}\right)$

$$\frac{d^2x_p}{dt^2} = -g\omega^2 e^{i\omega t} \qquad \frac{dx_p}{dt} = gi\omega e^{i\omega t}$$

Substituting gives $\quad -g\omega^2 + 2\gamma g i\omega + \omega_0^2 g = F$

$$g = \frac{F}{\omega_0^2 - \omega^2 + 2\gamma i\omega}$$

but $x_p = Re\left(g e^{i\omega t}\right)$

Convert to polar form: let $Z(\omega) = \omega_0^2 - \omega^2 + 2\gamma i\omega$

$$|Z| = \sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}$$

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

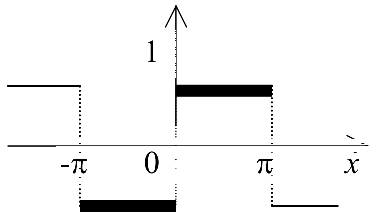
$$Z(\omega) = |Z|e^{i\phi}$$

$$x_p = Re\left[\frac{F}{|Z|e^{i\phi}}e^{i\omega t}\right]$$

$$x_p = Re\left[\frac{Fe^{-i\phi}}{|Z|}e^{i\omega t}\right] = Re\left[\frac{Fe^{i(\omega t-\phi)}}{|Z|}\right]$$

$$x_p = \frac{F}{|Z|}\cos(\omega t - \phi)$$

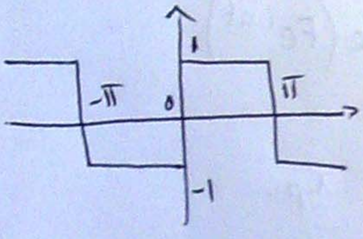
Week 5: Fourier Series



6. Show that the function shown has Fourier series

$$f(x) = \frac{4}{\pi}\sum_{n\,odd}\frac{1}{n}\sin nx$$

Explain why the series only contains sine terms.



is ODD so sine term only.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{n}\cos nx\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{1}{n}\cos nx\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left(\frac{1}{n} - \frac{1}{n}(-1)^n\right) + \frac{1}{\pi}\left(-\frac{1}{n}(-1)^n + \frac{1}{n}\right)$$

$$= \frac{2}{\pi n} - \frac{2}{\pi n}(-1)^n$$

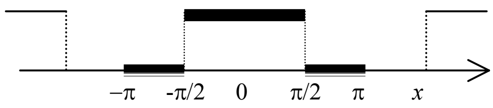
$= 0$ if n is even

$= \frac{4}{\pi n}$ if n is odd.

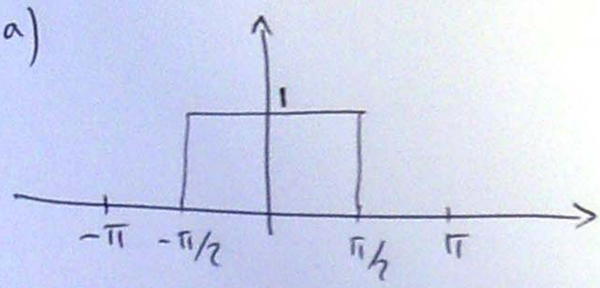
$$f(x) = \frac{4}{\pi}\sum_{n=odd}^{\infty}\frac{1}{n}\sin nx$$

7. (a) Show that the function shown has Fourier series

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \ldots\right)$$



⑦ a)



Series is even so cos terms only

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx \, dx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} dx = \frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \cos nx + 0$$

$$a_n = \frac{1}{\pi}\left[\frac{1}{n}\sin nx\right]_{-\pi/2}^{\pi/2} = \frac{1}{\pi}\left(\frac{1}{n}\sin\frac{n\pi}{2} - \frac{1}{n}\sin\left(-\frac{n\pi}{2}\right)\right)$$

$$= \frac{2}{\pi n}\left(\sin\frac{n\pi}{2}\right)$$

$n=1$ $a_n = \frac{2}{\pi}$

$n=2$ $a_n = 0$

$n=3$ $a_n = -\frac{2}{3\pi}$

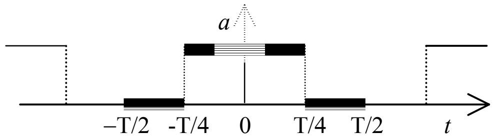
$n=4$ $a_n = 0$

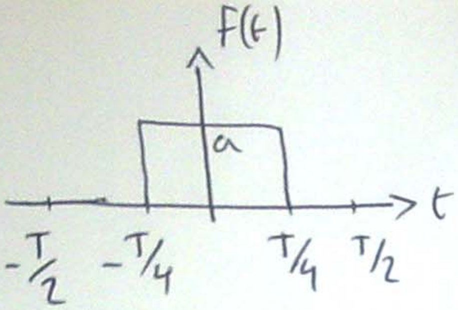
$n=5$ $a_n = \frac{2}{5\pi}$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\left(\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x + -\right)$$

(b) Hence deduce that the function below has series

$$f(t)=\frac{a}{2}+\frac{2a}{\pi}\left(\cos\frac{2\pi t}{T}-\frac{1}{3}\cos\frac{6\pi t}{T}+\frac{1}{5}\cos\frac{10\pi t}{T}-\ldots\right)$$





Mappings $\frac{x}{\pi}=\frac{2t}{T}$ and $\frac{f(x)}{1}=\frac{F(t)}{a}$

$x=\frac{2\pi t}{T}$ and $f(x)=\frac{F(t)}{a}$

$$f(x)=\frac{1}{2}+\frac{2}{\pi}\left(\cos x-\frac{1}{3}\cos 3x+\frac{1}{5}\cos 5x+\ldots\right)$$

$$\frac{F(t)}{a}=\frac{1}{2}+\frac{2}{\pi}\left(\cos\frac{2\pi t}{T}-\frac{1}{3}\cos\frac{6\pi t}{T}+\frac{1}{5}\cos\frac{10\pi t}{T}+\ldots\right)$$

$$f(t)=\frac{a}{2}+\frac{2a}{\pi}\left(\cos\frac{2\pi t}{T}-\frac{1}{3}\cos\frac{6\pi t}{T}+\frac{1}{5}\cos\frac{10\pi t}{T}+\ldots\right)$$

8. A function is defined on the range [0, π]: $f(x) = x \qquad 0 < x < \pi$

(a) Sketch the *odd extension* of this function. Show that the corresponding half-range sine series is

$$f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

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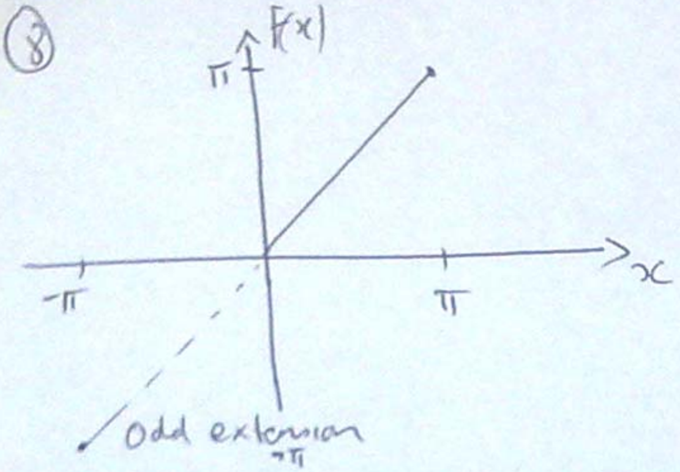

half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx.$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx$$

b_n by parts. $\int u\,dv = uv - \int v\,du$

$u = x \quad du = dx$

$dv = \sin nx \, dx$

$v = -\frac{1}{n} \cos nx$

$$b_n = \frac{2}{\pi}\left[-\frac{x}{n}\cos nx + \underbrace{\cancel{\frac{1}{n^2}\sin nx}}_{0}\right]_0^{\pi} = \frac{2}{\pi}\left(-\frac{\pi}{n}(-1)^n\right) = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

$$f(x) = 2\sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} \sin nx$$

(b) Sketch the *even extension* of this function. Find the corresponding half-range cosine series.

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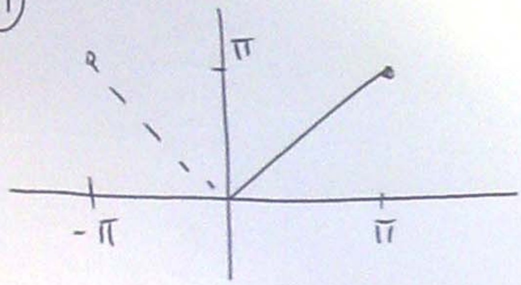

Even extension

half range Cosine Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx$$

a_n by parts $\int u\,dv = uv - \int v\,du$

$u = x,\ du = dx$

$dv = \cos nx\,dx$

$v = \frac{1}{n}\sin nx$

$$a_n = \frac{2}{\pi}\left[\frac{x}{n}\sin nx - \frac{1}{n^2}\cos nx\right]_0^{\pi} = \frac{2}{\pi}\left(-\frac{1}{n^2}(-1)^n - \frac{1}{n^2}\right) = \frac{2}{\pi n^2}\left((-1)^{n+1} - 1\right)$$

$= 0$ for n odd

$= -\frac{4}{\pi n^2}$ for n even

$$f(x) = \frac{\pi}{2} + \sum_{n=\text{even}} -\frac{4}{\pi n^2}\cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n\,\text{even}} \frac{1}{n^2}\cos nx$$