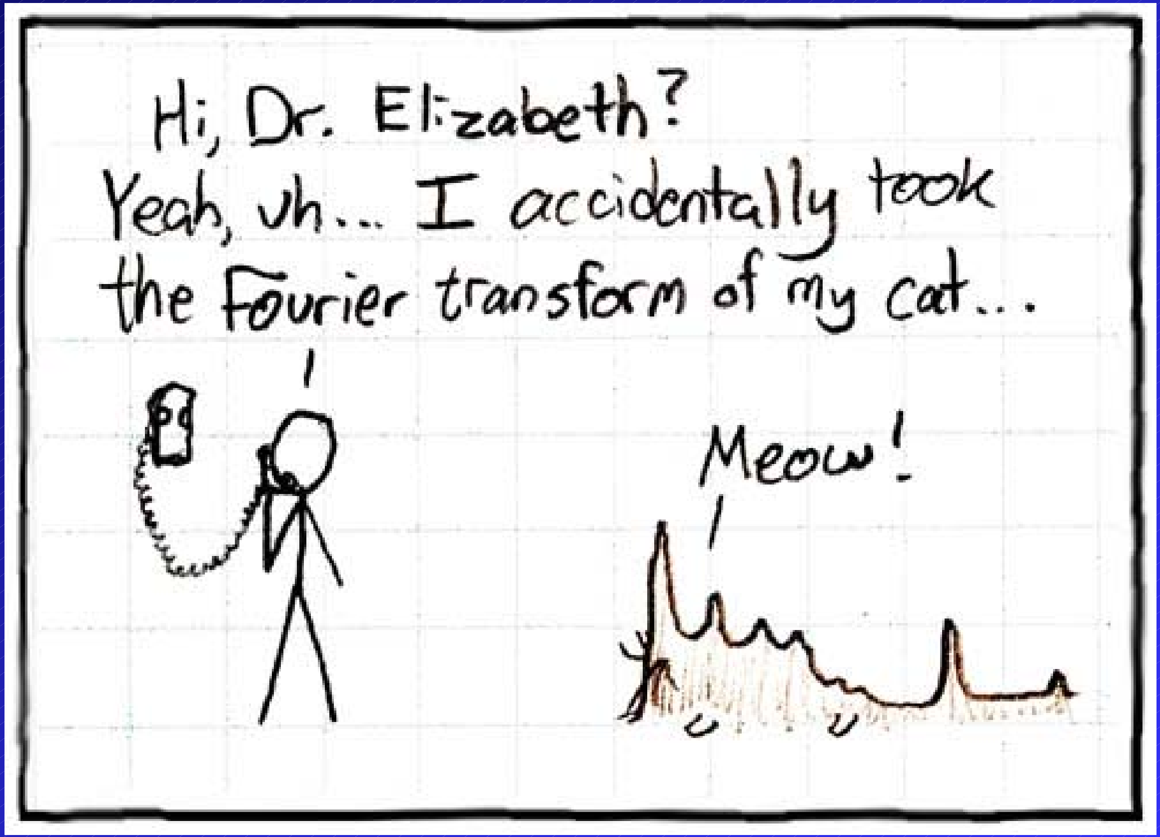
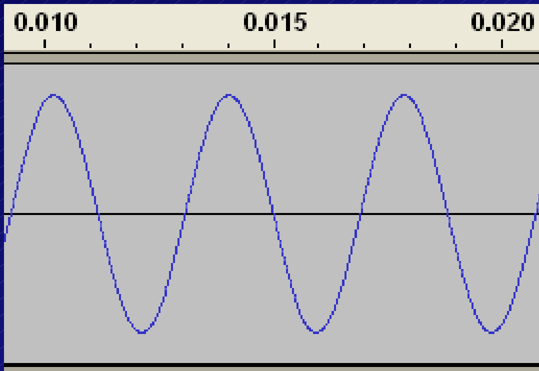
Topic 4 – Fourier Series

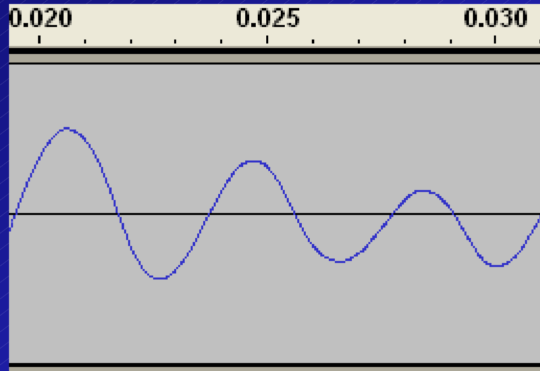


Waves – with repeating functions

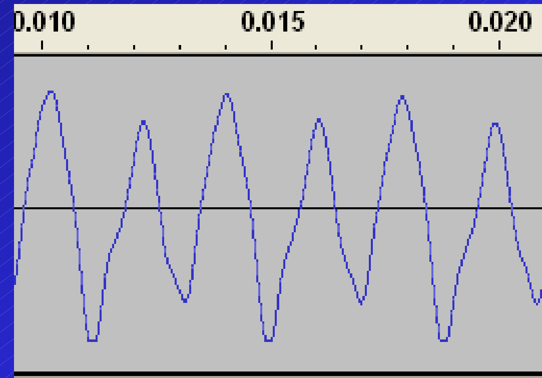
Signal generator



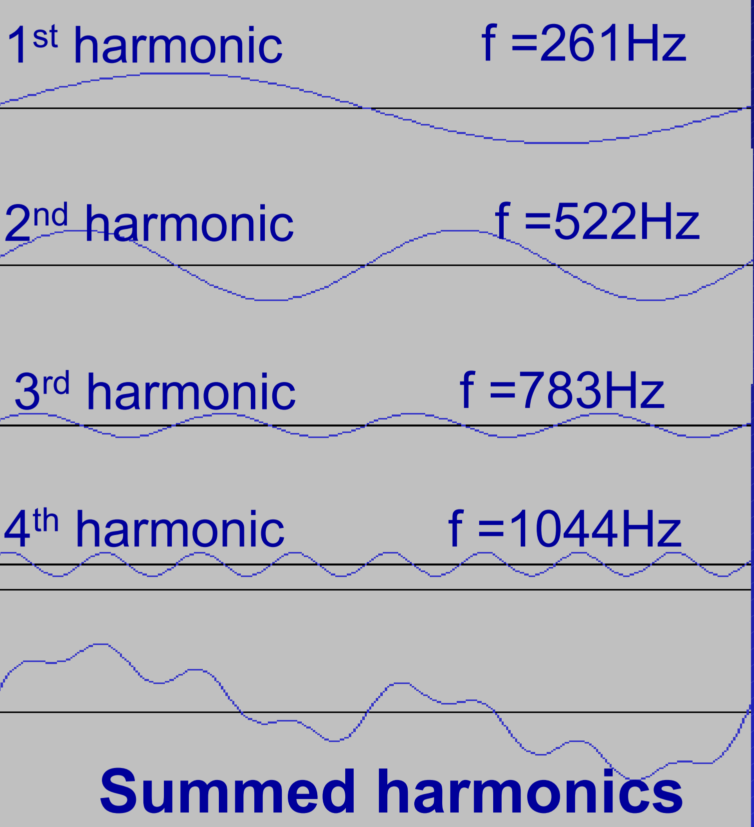
Classical guitar



Piano



Each instrument is playing a single note middle C (261Hz)



A single note will contain different fractions of each harmonic

$$y_1 = B_1 \sin \omega_1 t$$

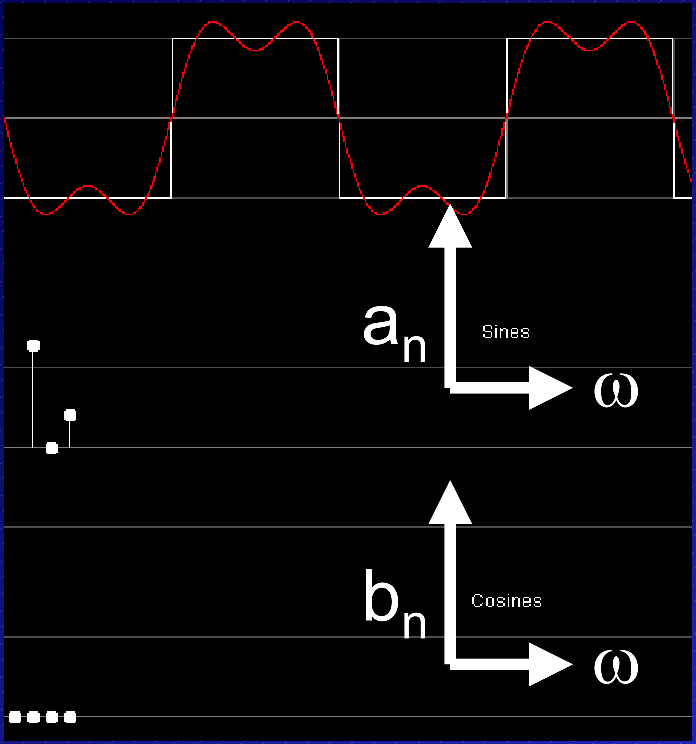
$$y_2 = B_2 \sin \omega_2 t$$

$$y_3 = B_3 \sin \omega_3 t$$

$$y_4 = B_4 \sin \omega_4 t$$

The summed output repeats with the period of the 1st harmonic

$$y_{total} = B_1 \sin \omega_1 t + B_2 \sin \omega_2 t + \ldots = \sum_{n=1}^{\infty} B_n \sin \omega_n t$$

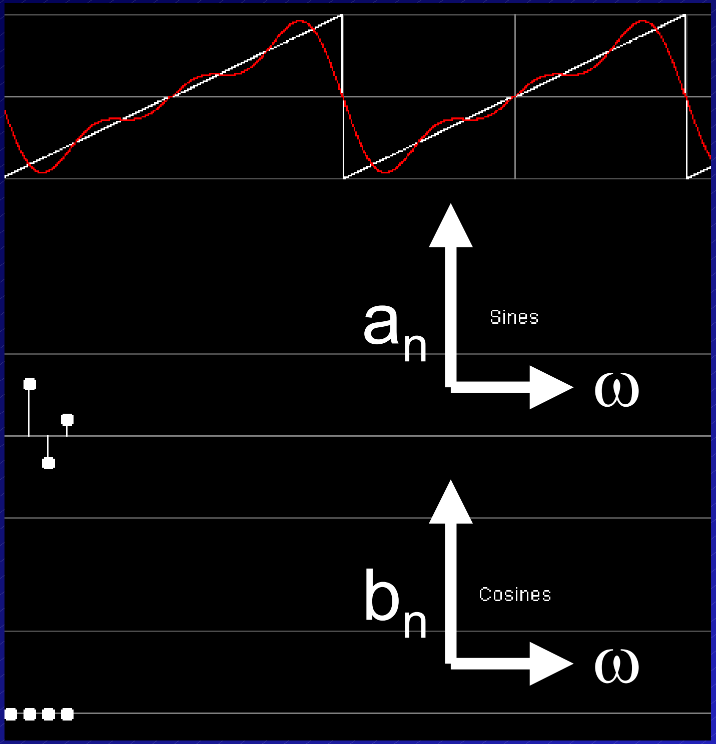


Fourier claimed that any repeating pattern could be represented by a summed series of cosine and sine terms

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\omega_n t + b_n \sin\omega_n t$$

http://www.univie.ac.at/future.media/moe/galerie/fourier/fourier.html

http://www.falstad.com/fourier/



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Fourier said

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_n t + b_n \sin \omega_n t$$

The pattern will repeat with period of the1st harmonic frequency

$\omega = \omega_1$ so since

$$\omega_n = n\omega \quad and \quad \omega = 2\pi f \quad and \quad T = \frac{1}{f}, \; then \quad \omega_n = \frac{2n\pi}{T}$$

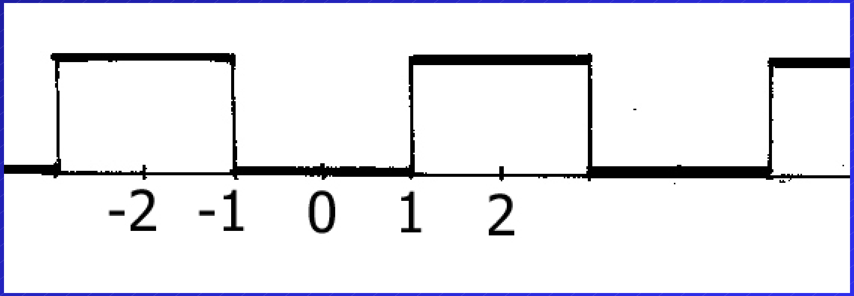
$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{2n\pi t}{T} + b_n \sin \frac{2n\pi t}{T}$$

where T is the period of the repeating function

Or if the pattern repeats with period L where L is a distance

we can write $$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

Here period is $L = 4$



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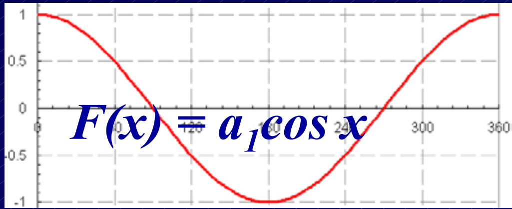
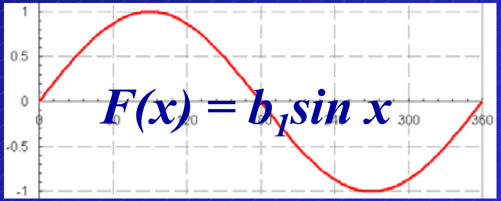
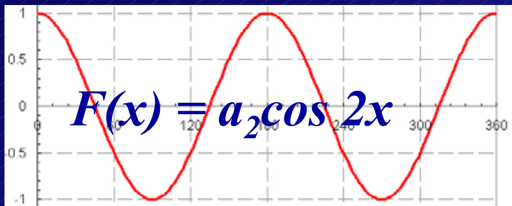
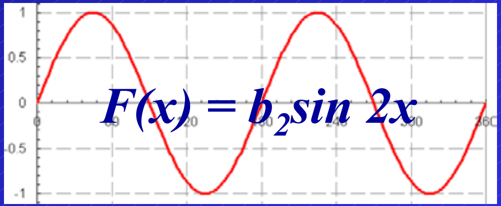


n = 1

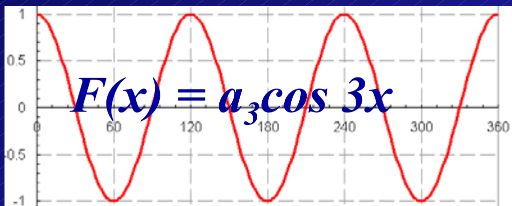




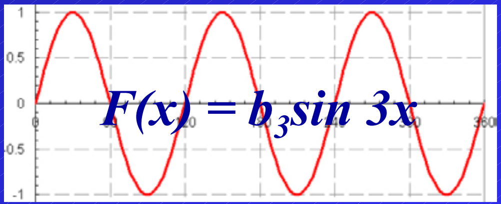
n = 2



For these cases L is taken to be 2π to simplify expressions



n = 3

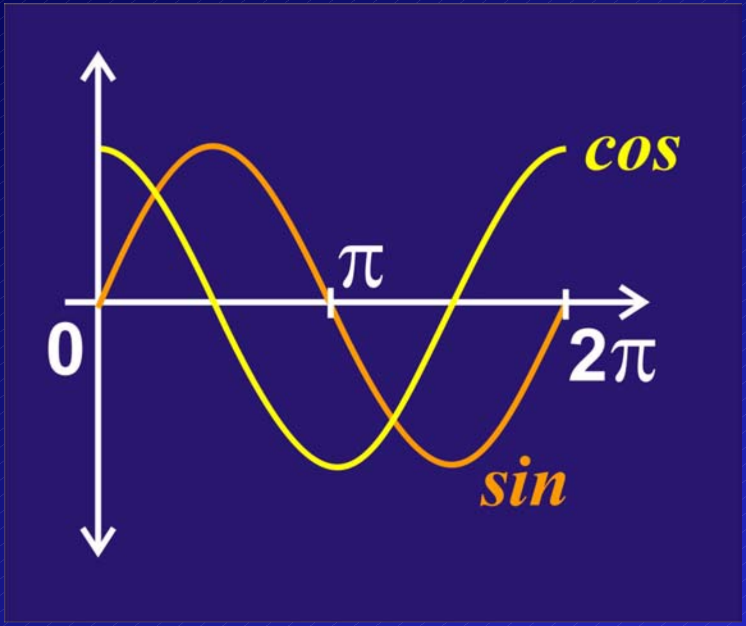


Need to find a_0, a_n and b_n

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

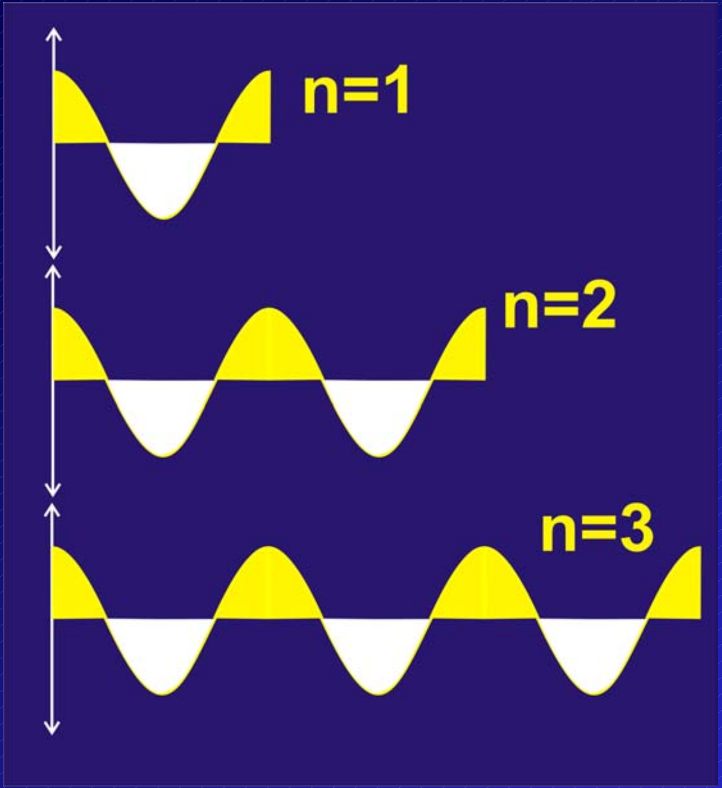
… some background work required ….

Background I - Integrating over 2π



$$\int_0^{2\pi} \cos x \; dx = 0$$

$$\int_0^{2\pi} \sin x \; dx = 0$$

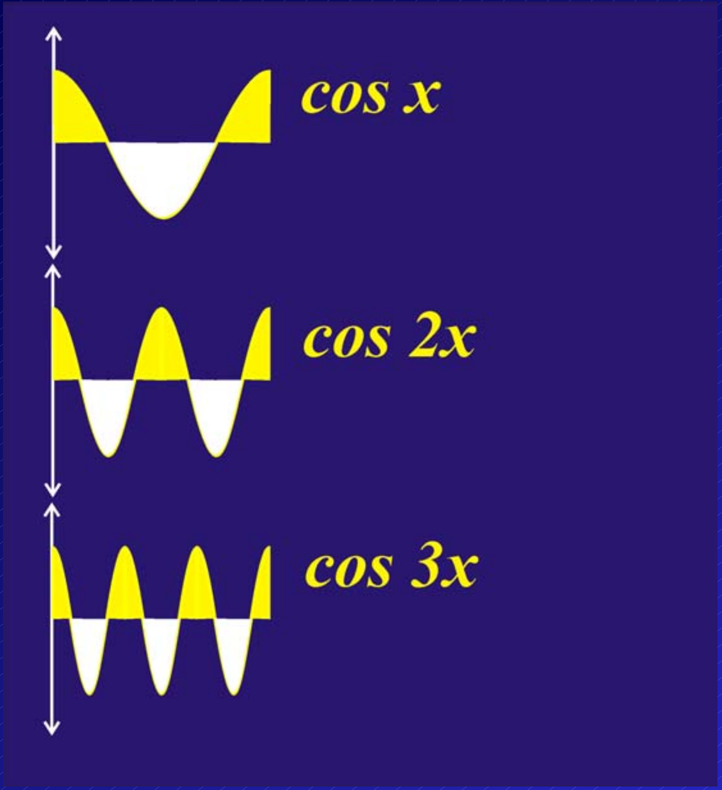


Integrating over 2πn

$$\int_0^{2\pi n} \cos x \, dx = 0 \quad for \; all \;\; n$$

$$\int_0^{2\pi n} \sin x \, dx = 0 \quad for \; all \;\; n$$

Integrating over 2π



$$\int_0^{2\pi} \cos nx \; dx = 0 \quad for \;\; all \;\; n$$

$$\int_0^{2\pi} \sin nx \; dx = 0 \quad for \;\; all \;\; n$$

Background II - useful integrals

$$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}$$

$$\cos A \cos B = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)$$

1 $$\int_0^{2\pi} \cos^2 nx \; dx = \int_0^{2\pi}\left(\frac{1}{2}\cos 2nx + \frac{1}{2}\right) dx = \left[\frac{1}{4n}\sin 2nx + \frac{x}{2}\right]_0^{2\pi} = \pi$$

2 $$\int_0^{2\pi} \cos nx \cos mx \; dx = \int_0^{2\pi}\frac{1}{2}\{\cos(n-m)x\}dx + \int_0^{2\pi}\frac{1}{2}\{\cos(n+m)x\}\,dx = 0$$

$n \neq m$

$$\int_0^{2\pi} \sin nx \; dx = 0 \quad for \;\; all \quad n$$

$$\int_0^{2\pi} \cos nx \; dx = 0 \quad for \;\; all \quad n$$

$$\int_0^{2\pi} \cos nx \cos mx \; dx = \int_0^{2\pi} \sin nx \sin mx \; dx = 0 \quad for \;\; all \quad m \neq n$$

$$\int_0^{2\pi} \sin nx \cos mx \; dx = 0 \quad for \;\; all \quad m \;\; and \;\; n$$

Remember: **odd x even = odd**

$$\int_0^{2\pi} \sin^2 nx \; dx = \int_0^{2\pi} \cos^2 nx \; dx = \pi \quad for \;\; all \quad n$$

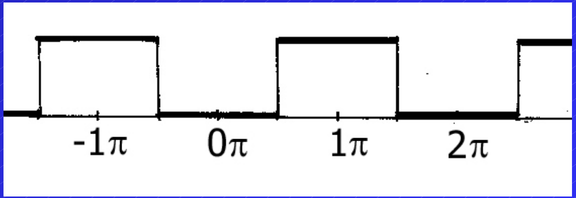
Finding coefficients of the Fourier Series…a_0

Remember how the Fourier series can be written like this for a period L

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

For simplicity let's make L = 2π so we can write…

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$



Finding coefficients of the Fourier Series…a_0

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Repeat period 2π

Take this equation and integrate both sides over a period

$$\int_0^{2\pi} f(x)dx = \frac{1}{2}a_0 \int_0^{2\pi} dx + \sum_{n=1}^{\infty}\left[a_n \int_0^{2\pi} \cos nx\, dx + b_n \int_0^{2\pi} \sin nx\, dx \right]$$

Clearly on the RHS the only non-zero term is the a_0 term

$$\int_0^{2\pi} f(x)dx = \frac{1}{2}a_0 \int_0^{2\pi} dx = \frac{1}{2}a_0(2\pi - 0) = \pi a_0$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

Finding coefficients of the Fourier Series…a_n

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

This time multiply both sides by ***cos(x)*** and integrate over a period

$$\int_0^{2\pi} f(x)\cos x dx = \frac{1}{2}a_0 \int_0^{2\pi} \cos x dx + \sum_{n=1}^{\infty}\left[a_n \int_0^{2\pi} \cos nx \cos x \, dx + b_n \int_0^{2\pi} \sin nx \cos x \, dx \right]$$

Finding coefficients of the Fourier Series…a_1

$$\int_0^{2\pi} f(x)\cos x\,dx = \frac{1}{2}a_0\int_0^{2\pi} \cos x\,dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx\cos x\,dx + b_n\int_0^{2\pi}\sin nx\cos x\,dx\right]$$

On RHS, only the a_1 term survives as it is only term where n=1 (Orthogonality)

$$\int_0^{2\pi} f(x)\cos x\,dx = a_1\int_0^{2\pi}\cos x\cos x\,dx = a_1\int_0^{2\pi}\cos^2 x\,dx = a_1\pi$$

Hence we find

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos x\,dx$$

Finding coefficients of the Fourier Series…a_n

To find all coefficients a_n multiply both sides of the Fourier series by $\cos(mx)$, then integrate over a period:

$$\int_0^{2\pi} f(x)\cos mx dx = \frac{1}{2}a_0\int_0^{2\pi} \cos mx dx + \sum_{n=1}^{\infty}\left[a_n\int_0^{2\pi}\cos nx \cos mx\, dx + b_n\int_0^{2\pi}\sin nx \cos mx\, dx\right]$$

On the RHS, only the $m = n$ term survives the integration

$$\int_0^{2\pi} f(x)\cos mx dx = a_m\int_0^{2\pi}\cos^2 mx\, dx = a_m\pi$$

$$a_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx dx$$

Coefficients of the Fourier Series…

In a similar way, multiplying both sides of the Fourier series by $\sin(mx)$, then integrating over a period we get:

$$b_m = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx$$

Coefficients of the Fourier Series

The Fourier series can be written with period 2π as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

The Fourier series coefficients can be found by:-

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

Coefficients of the Fourier Series

The Fourier series can be written with period L as

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

The Fourier series coefficients can be found by:-

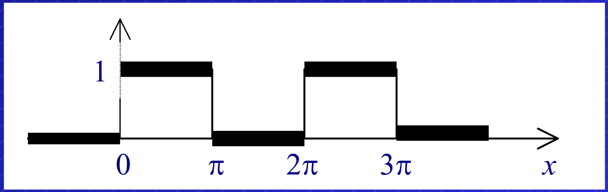
$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

Finding the coefficients of a Fourier Series

Step 1.
Write down the function *f(x)* in terms of *x*.
What is the period?



Step 2.
Use equation to find a_0?

$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

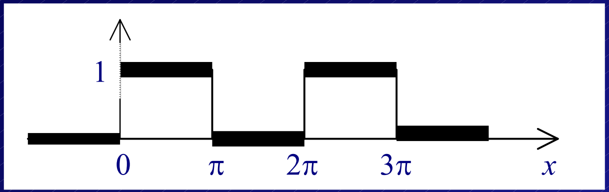
Step 3.
Use equation to find a_n?

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx$$

Step 4.
Use equation to find b_n?

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

Example 4.1 - page 35



1. The function *f(x)?* What's the period?

$$f(x)=\begin{cases}1 & 0<x<\pi\\ 0 & \pi<x<2\pi\end{cases}$$

Period is 2π

2. Use equation to find a_0?

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)dx=\frac{1}{\pi}\int_0^{\pi} 1dx+\frac{1}{\pi}\int_{\pi}^{2\pi} 0dx=\frac{1}{\pi}[x]_0^{\pi}=1$$

3. Use equation to find a_n?

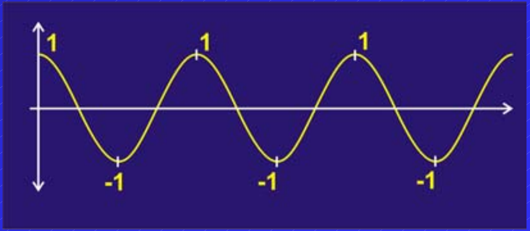
$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}(1)\cos nx\,dx+\frac{1}{\pi}\int_{\pi}^{2\pi}(0)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}\cos nx\,dx=\frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}=0$$

4. Use equation to find b_n?

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(0)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$b_n = \frac{-1}{\pi}\left(\frac{\cos n\pi}{n} - \frac{\cos 0}{n}\right) = \frac{-1}{\pi}\left(\frac{\cos n\pi}{n} - \frac{1}{n}\right) = \frac{-1}{\pi n}\left((-1)^n - 1\right)$$



Step 5. Write out values of b_n for n = 1, 2, 3, 4, 5, ….

$$L = 2\pi$$

$$a_0 = 1$$

$$a_n = 0$$

$$b_n = \frac{1}{n\pi}\left(1-(-1)^n\right)$$

$$b_1 = \frac{1}{\pi}\left(1-(-1)\right) = \frac{2}{\pi}$$

$$b_1 = \frac{1}{2\pi}\left(1-(1)\right) = 0$$

$$b_1 = \frac{1}{3\pi}\left(1-(-1)\right) = \frac{2}{3\pi}$$

$$b_1 = \frac{1}{4\pi}\left(1-(1)\right) = 0$$

$$b_5 = \frac{1}{5\pi}\left(1-(-1)\right) = \frac{2}{5\pi}$$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

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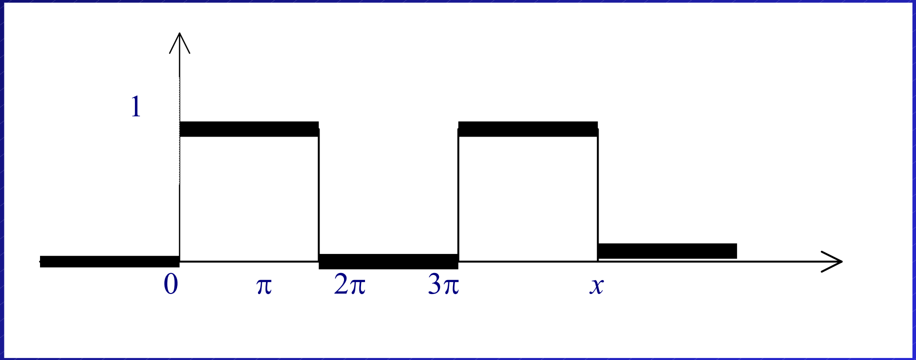
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \ldots$$

So what does the Fourier series look like if we only use first few terms?



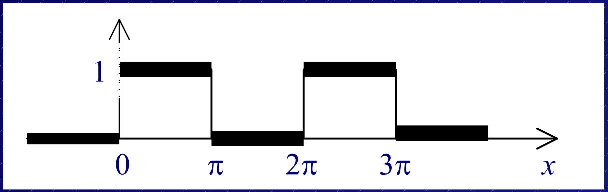
Lecture 7 – Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

- **More examples of Fourier series**
- **Describing pulses with Fourier series**

Example 4.1 - page 35



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4. Use equation to find b_n?

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi}(1)\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(0)\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

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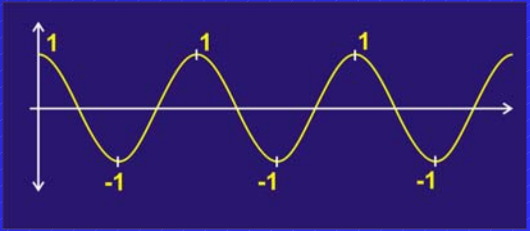

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$$b_1 = \frac{1}{4\pi}\left(1-(1)\right) = 0$$

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$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \ldots$$

Fourier Series - QUIZ

1. What is $1+(-1)^n$ when n = 3 ? $1+(-1)^3 = 0$

2. What is $1+(-1)^n$ when n = 52 ? $1+(-1)^{52} = 2$

3. What is $1+(\cos 2\pi n)$ when n = 1 ? $1+(1) = 2$

4. What is $1+(\cos 2\pi n)$ when n = 17 ? $1+(1) = 2$

5. What is $1+(\cos 2\pi n)$ when n = 52 ? $1+(1) = 2$

6. What is $1+(\cos \pi n)$ when n = 1 ? $1+(\cos \pi) = 1+(-1) = 0$

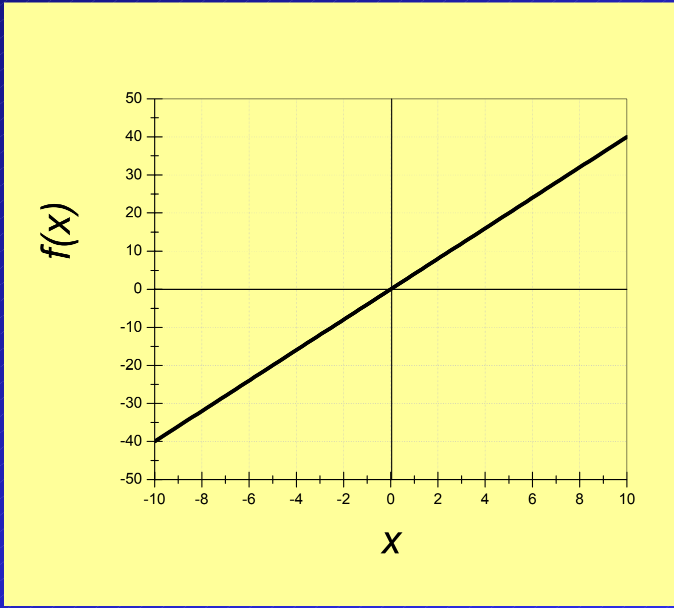
7. What is $1+(\cos \pi n)$ when n = 4 ? $1+(\cos 4\pi) = 1+(1) = 2$

Fourier Series - QUIZ

8. What is

$$I = \int_{-10}^{10} 4x \; dx$$

$$I = \int_{-10}^{10} 4x \; dx = \left[2x^2\right]_{-10}^{10} = 200 - 200 = 0$$

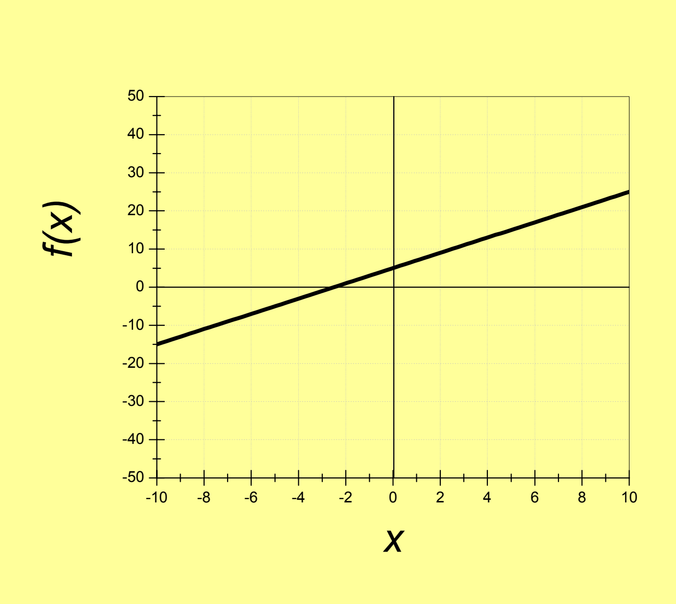


Fourier Series - QUIZ

9. What is

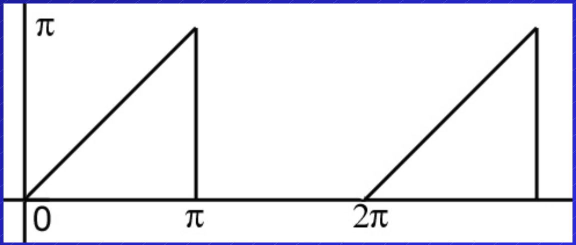
$$I = \int_{-10}^{10} (2x+5)\ dx$$

$$I = \int_{-10}^{10} (2x+5)\ dx = \left[x^2 + 5x\right]_{-10}^{10} = 100$$



Finding coefficients of the Fourier Series

Find Fourier series to represent this repeat pattern.



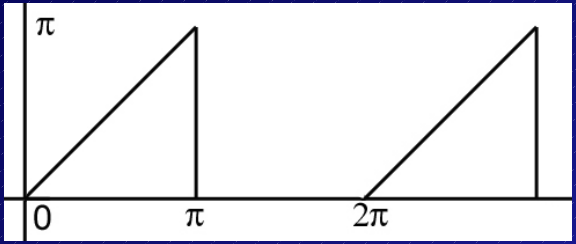
Steps to calculate coefficients of Fourier series

1. Write down the function *f(x)* in terms of *x*. What is period?

$$f(x)=\begin{cases} x & 0<x<\pi \\ 0 & \pi \le x < 2\pi \end{cases}$$

Period is 2π

Finding coefficients of the Fourier Series



$$f(x)=\begin{cases} x & 0<x<\pi \\ 0 & \pi \le x < 2\pi \end{cases}$$

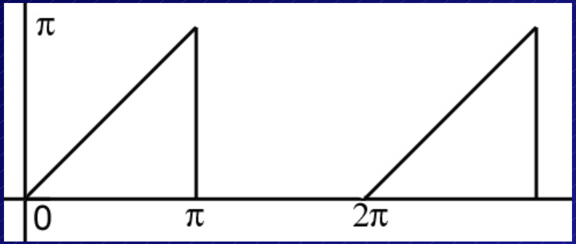
$$a_0 = \frac{2}{L}\int_0^L f(x)dx$$

Steps to calculate coefficients of Fourier series

2. Use equation to find a_0?

$$a_0 = \frac{2}{2\pi}\int_0^{2\pi} f(x)dx = \frac{1}{\pi}\int_0^{\pi} xdx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

Finding coefficients of the Fourier Series



$$f(x)=\begin{cases} x & 0<x<\pi \\ 0 & \pi\le x<2\pi \end{cases}$$

First 5 terms (*n=1 to 5*)

2. Use equation to find a_n?

$$a_n=\frac{1}{\pi}\int_0^L f(x)\cos nx\,dx$$

Left side - find coefficients a_n

3. Use equation to find b_n?

$$b_n=\frac{1}{\pi}\int_0^L f(x)\sin nx\,dx$$

Right side - find coefficients b_n

$$f(x)=\begin{cases} x & 0<x<\pi \\ 0 & \pi \le x < 2\pi \end{cases}$$

a

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\cos nx\,dx$$

Integrate by parts $\int u\,dv = uv - \int v\,du$

$$u = x$$

$$du = dx$$

$$dv = \cos nx\,dx$$

$$v = \int \cos nx\,dx = \frac{1}{n}\sin nx$$

Integrate by parts $\int u\,dv = uv - \int v\,du$

$$u = x$$

$$du = dx$$

$$dv = \cos nx\,dx$$

$$v = \int \cos nx\,dx = \frac{1}{n}\sin nx$$

$$a_n = \left[\frac{x}{\pi n}\sin nx\right]_0^{\pi} - \frac{1}{\pi}\int_0^{\pi} \frac{1}{n}\sin nx\,dx$$

$$a_n = \left[\frac{x}{\pi n}\sin nx\right]_0^{\pi} + \left[\frac{1}{\pi n^2}\cos nx\right]_0^{\pi}$$

$$a_n = \left(\frac{1}{n}\sin n\pi + \frac{1}{\pi n^2}\cos n\pi\right) - \left(0 + \frac{1}{\pi n^2}\right)$$

$$a_n = \left(\frac{1}{n}\sin n\pi + \frac{1}{\pi n^2}\cos n\pi\right) - \left(\frac{1}{\pi n^2}\right) = \left(\frac{1}{\pi n^2}(-1)^n\right) - \left(\frac{1}{\pi n^2}\right) = \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

$$a_n = \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

$$a_1 = \left(-\frac{1}{\pi}\right) - \left(\frac{1}{\pi}\right) = -\frac{2}{\pi}$$

$$a_2 = \left(\frac{1}{4\pi}\right) - \left(\frac{1}{4\pi}\right) = 0$$

$$a_3 = \left(-\frac{1}{9\pi}\right) - \left(\frac{1}{9\pi}\right) = -\frac{2}{9\pi}$$

$$a_4 = \left(\frac{1}{16\pi}\right) - \left(\frac{1}{16\pi}\right) = 0$$

$$a_5 = \left(-\frac{1}{25\pi}\right) - \left(\frac{1}{25\pi}\right) = -\frac{2}{25\pi}$$

$$f(x) = \begin{cases} x & 0 < x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$$

b

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx\, dx + \frac{1}{\pi}\int_\pi^{2\pi} 0\sin nx\, dx$$

Integrate by parts $\int u\,dv = uv - \int v\,du$

$$u = x$$

$$du = dx$$

$$dv = \sin nx\,dx$$

$$v = \int \sin nx\,dx = -\frac{1}{n}\cos nx$$

Integrate by parts $\int u\,dv = uv - \int v\,du$

$$u = x$$

$$du = dx$$

$$dv = \sin nx\,dx$$

$$v = \int \sin nx\,dx = -\frac{1}{n}\cos nx$$

$$b_n = -\frac{1}{\pi}\left[\frac{x}{n}\cos nx\right]_0^{\pi} + \frac{1}{\pi}\int_0^{\pi}\frac{1}{n}\cos nx\,dx$$

$$b_n = \left[-\frac{x}{\pi n}\cos nx\right]_0^{\pi} + \left[\frac{1}{\pi n^2}\sin nx\right]_0^{\pi}$$

$$b_n = \left(-\frac{1}{n}\cos n\pi + \frac{1}{\pi n^2}\sin n\pi\right)$$

$$b_n = \left(-\frac{1}{n}\cos n\pi + \frac{1}{\pi n^2}\sin n\pi \right)$$

$$b_n = -\frac{1}{n}(-1)^n$$

$$b_1 = 1$$

$$b_2 = -\frac{1}{2}$$

$$b_3 = \frac{1}{3}$$

$$b_4 = -\frac{1}{4}$$

$$b_5 = \frac{1}{5}$$

$$a_0 = \frac{\pi}{2}$$

$$a_1 = -\frac{2}{\pi}$$

$$a_2 = 0$$

$$a_3 = -\frac{2}{9\pi}$$

$$a_4 = 0$$

$$a_5 = -\frac{2}{25\pi}$$

$$b_1 = 1$$

$$b_2 = -\frac{1}{2}$$

$$b_3 = \frac{1}{3}$$

$$b_4 = -\frac{1}{4}$$

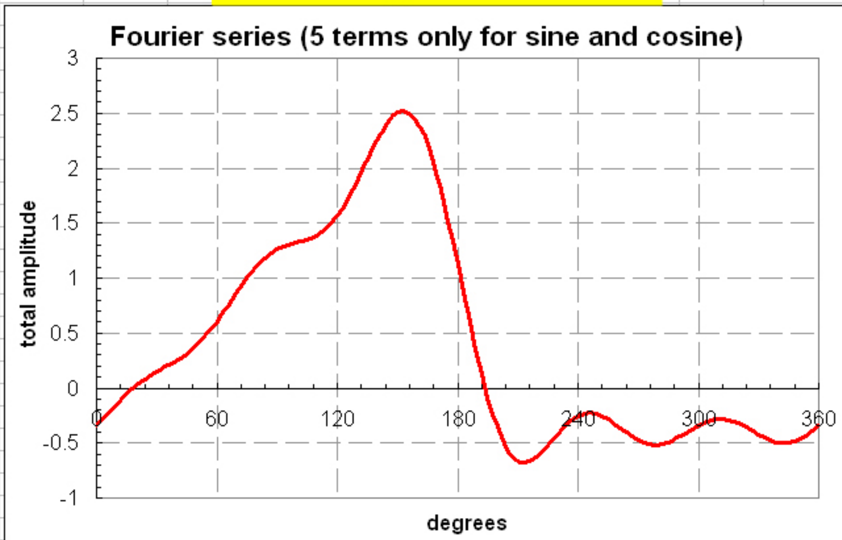
$$b_5 = \frac{1}{5}$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\cos x - \frac{2}{9\pi}\cos 3x - \frac{2}{25\pi}\cos 5x + \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x + \ldots.$$

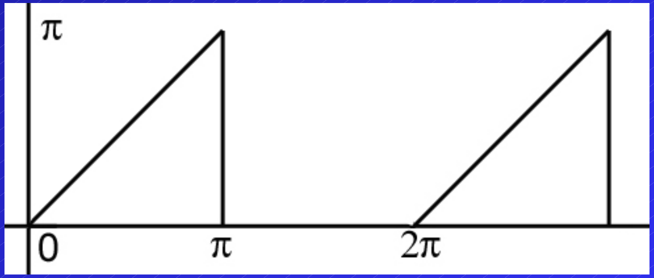
$$f(x)=\frac{\pi}{4}-\frac{2}{\pi}\cos 1x-\frac{2}{9\pi}\cos 3x-\frac{2}{25\pi}\cos 5x+1\sin 1x-\frac{1}{2}\sin 2x+\frac{1}{3}\sin 3x-\frac{1}{4}\sin 4x+\frac{1}{5}\sin 5x+....$$

amplitudes of the cosine harmonics						amplitudes of the sine harmonics				
value of a0	value of a1	value of a2	value of a3	value of a4	value of a5	value of b1	value of b2	value of b3	value of b4	value of b5
0.785	-0.63694	0	-0.07077	0	-0.025478	1	-0.5	0.333	-0.25	0.2

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx+b_n\sin nx$$

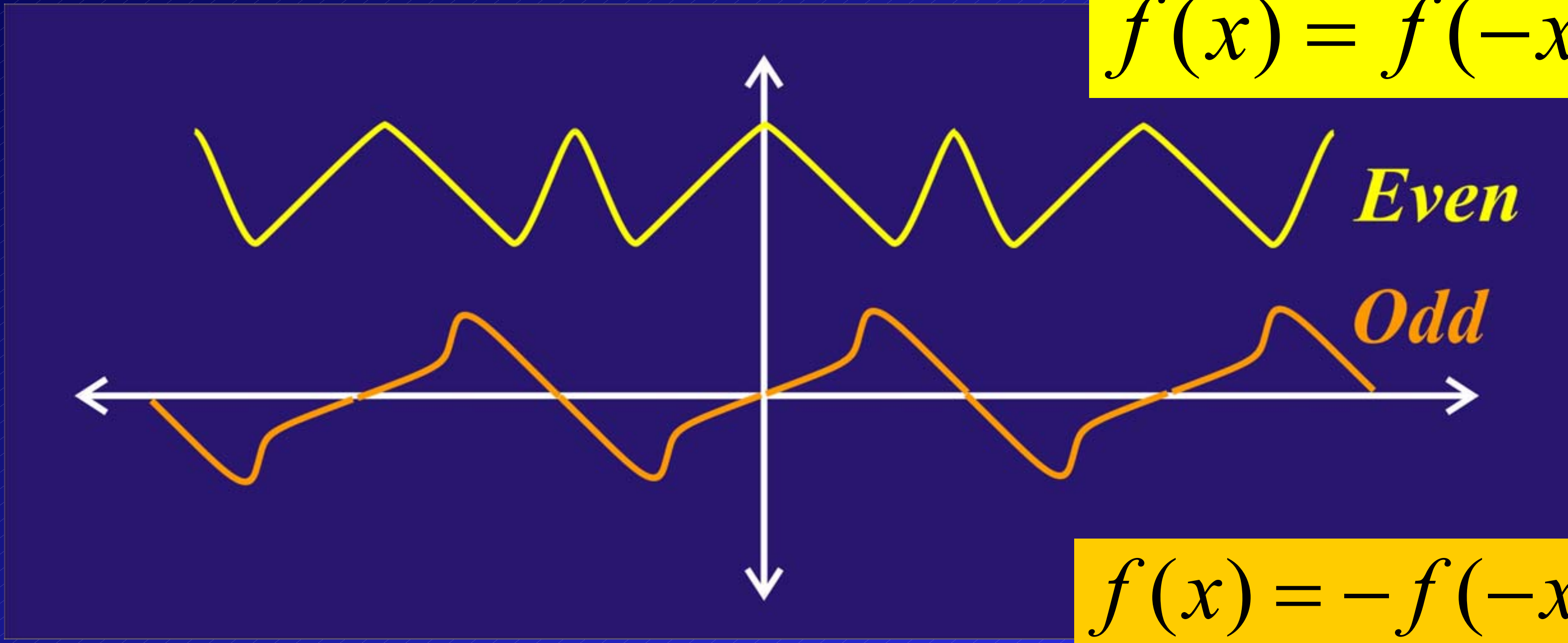


Check our Fourier series using Fourier_checker.xls

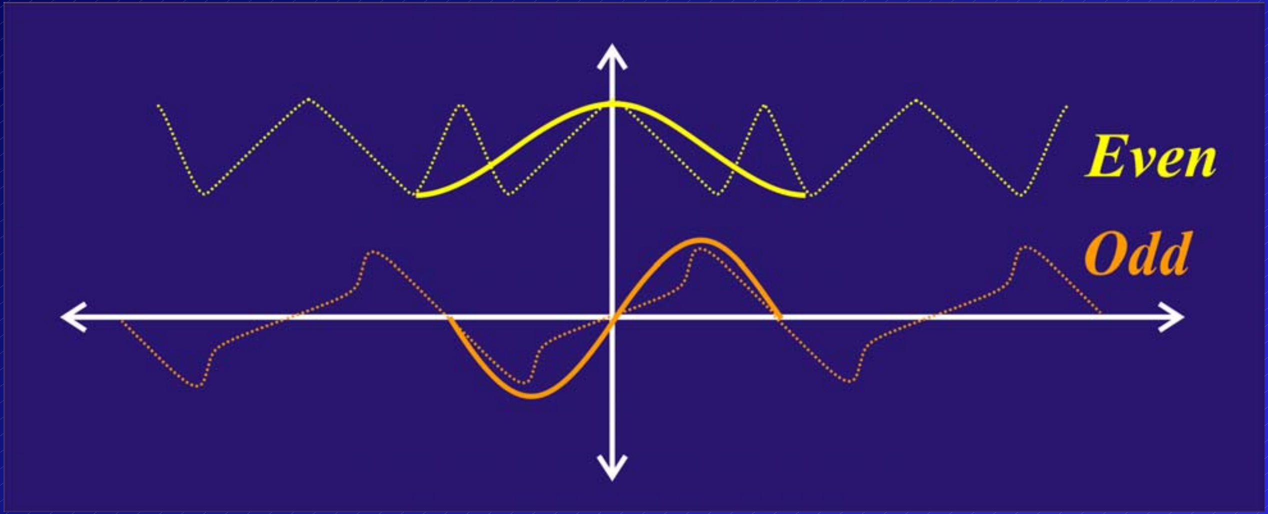


Fourier Series of even and odd repeating functions

$$f(x) = f(-x)$$



$$f(x) = -f(-x)$$

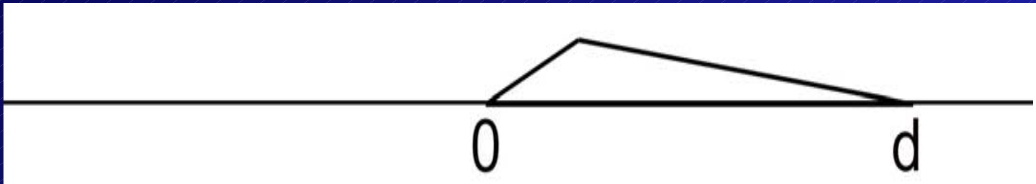


Only *sine* terms required to define an **odd function**

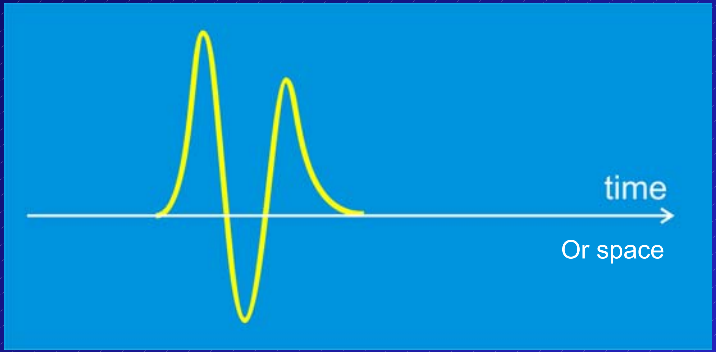
Only *cosine* terms required to define an **even function**

Only an even function can have an offset.

Fourier Series applied to pulses



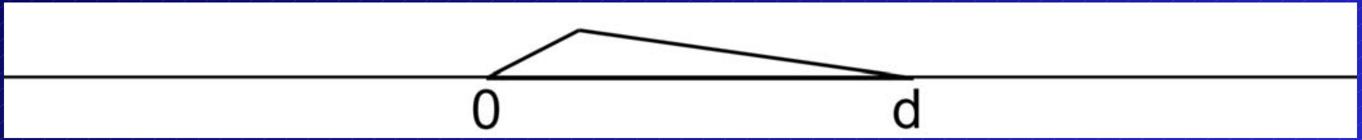
$$f(x) = \begin{cases} = 0 & for\ x < 0 \\ \neq 0 & for\ 0 \leq x \leq d \\ = 0 & for\ x > d \end{cases}$$



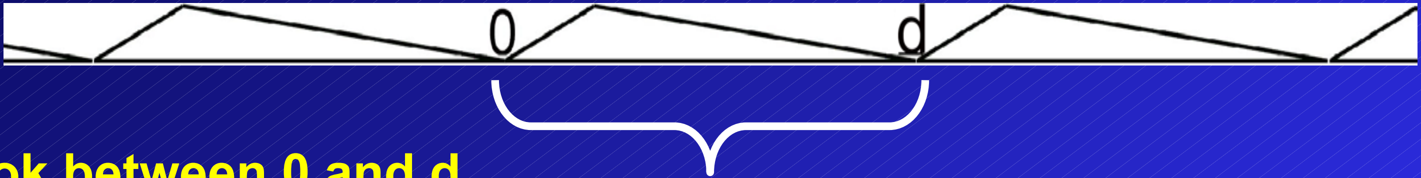
Laser light pulse

Initial displacement of a guitar string

Electronic wavefunction of a molecule

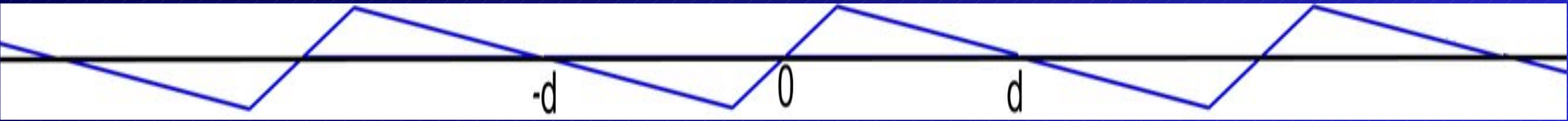


becomes

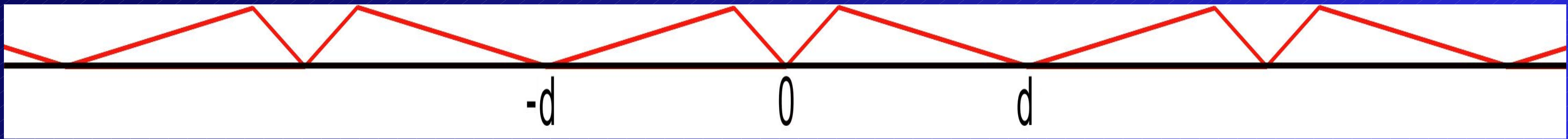


but only look between 0 and d

This approach is fine but it leads to a lot of work in the integration stage.



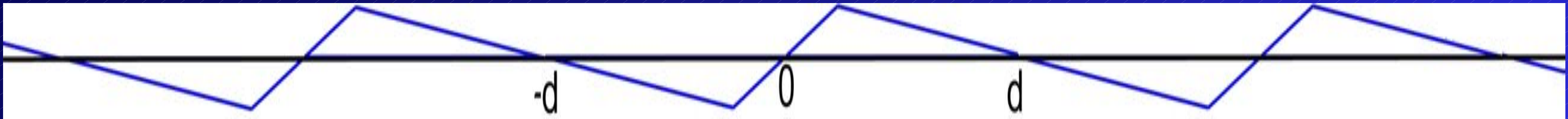
Odd function (only sine terms)



Even function (only cosine terms)

What is period of the repeating pattern now?

Half-range sine series – where $L=2d$



We saw earlier that for a function with period *L* the Fourier series is:-

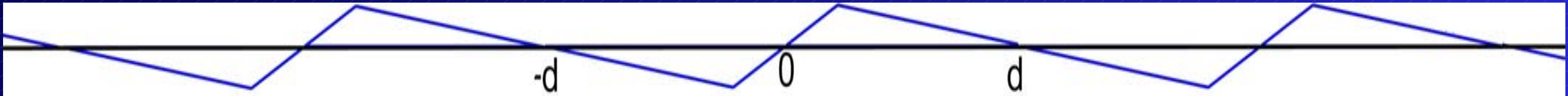
$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos\frac{2n\pi x}{L}+b_n\sin\frac{2n\pi x}{L}$$

where

$$a_n=\frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx \qquad b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

In the half range case we have a function of **period *2d*** which is odd and so contains only sine terms

Half-range sine series – where $L=2d$



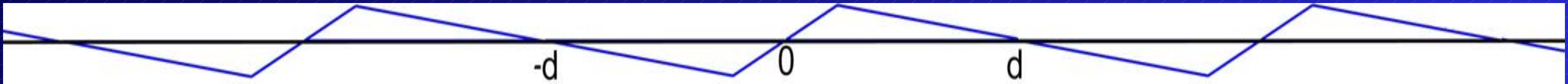
In the half range case we have a function of **period 2d** which is odd and so contains only sine terms

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{2n\pi x}{(2d)}=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{d}$$

where $$b_n=\frac{2}{2d}\int_{-d}^{d} f(x)\sin\frac{2n\pi x}{(2d)}\,dx=\frac{1}{d}\int_{-d}^{d} f(x)\sin\frac{n\pi x}{d}\,dx$$

The series is valid only between x =0 and x = d

Half-range sine series



$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{d}$$

$$b_n = \frac{1}{d}\int_{-d}^{d} f(x)\sin\frac{n\pi x}{d}\,dx$$

$$f(x) = ODD$$

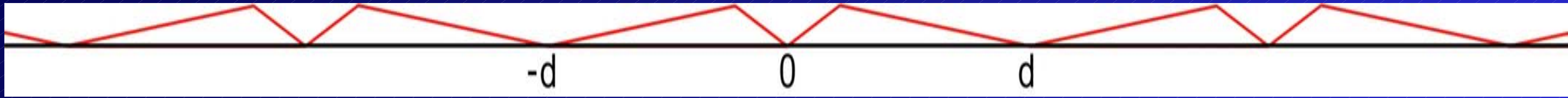
$$\sin x = ODD$$

$$ODD \times ODD = EVEN$$

$$\int_{-d}^{d} EVEN = 2\int_{0}^{d} EVEN$$

$$b_n = \frac{2}{d}\int_{0}^{d} f(x)\sin\frac{n\pi x}{d}\,dx$$

Half-range cosine series



For a function with period *L* the Fourier series is:-

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

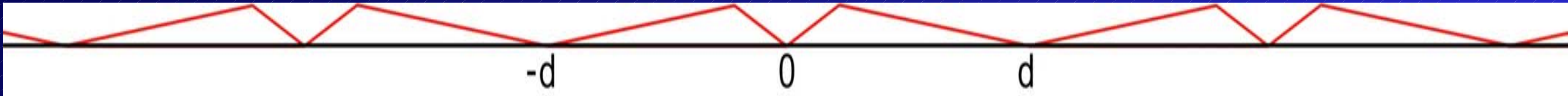
where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx$$

We have a function of period *2d* but this time it is even and so contains only cosine terms

Half-range cosine series



We have a function of period *2d* but this time it is even and so contains only cosine terms

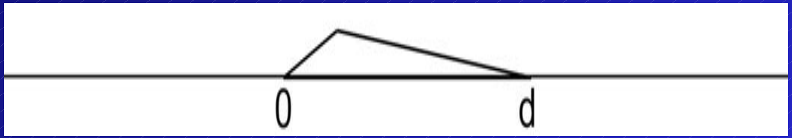
$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{(2d)} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{d}$$

$$f(x) = EVEN$$
$$\cos x = EVEN$$

$$\int_{-d}^{d} EVEN = 2\int_{0}^{d} EVEN$$

$$a_n = \frac{2}{2d}\int_{-d}^{d} f(x)\cos\frac{2n\pi x}{(2d)}dx = \frac{1}{d}\int_{-d}^{d} f(x)\cos\frac{n\pi x}{d}dx = \frac{2}{d}\int_{0}^{d} f(x)\cos\frac{n\pi x}{d}dx$$

The Fourier series for a pulse such as



can be written as either a half range sine or cosine series. However the series is only valid between 0 and d

Half range sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{d}$$

where

$$b_n = \frac{2}{d} \int_0^d f(x) \sin \frac{n\pi x}{d} dx$$

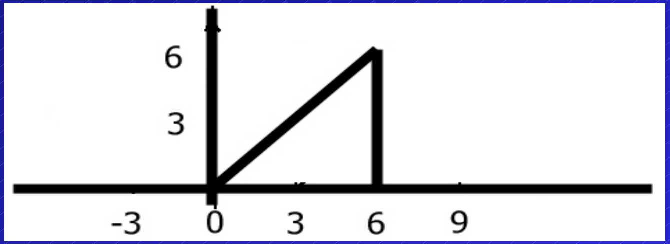
Half range cosine series

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{d}$$

where

$$a_n = \frac{2}{d} \int_0^d f(x) \cos \frac{n\pi x}{d} dx$$

Find Half Range ***Sine*** Series which represents the displacement ***f(x)***, between x = 0 and 6, of the pulse shown to the right



The pulse is defined as $f(x) = x \quad for\ 0 < x \leq 6$ with a length d = 6

So $b_n = \frac{2}{d}\int_0^d f(x)\sin\frac{n\pi x}{d}dx = \frac{2}{6}\int_0^6 x\sin\frac{n\pi x}{6}dx$ Integrate by parts $\int udv = uv - \int vdu$

$$b_n = \frac{2}{d}\int_0^d f(x)\sin\frac{n\pi x}{d}dx = \frac{2}{6}\int_0^6 x\sin\frac{n\pi x}{6}dx$$

Integrate by parts

$$\int u\,dv = uv - \int v\,du$$

So $so\ set\ u = x\ and\ \sin\frac{n\pi x}{6}dx = dv$

$$v = \int \sin\frac{n\pi x}{6}dx = -\frac{6}{n\pi}\cos\frac{n\pi x}{6} \quad and \quad du = dx$$

$$b_n = \frac{1}{3}\left(\left[-\frac{6x}{n\pi}\cos\frac{n\pi x}{6}\right]_0^6 + \int_0^6 \frac{6}{n\pi}\cos\frac{n\pi x}{6}\right) = \frac{1}{3}\left(\left[-\frac{6x}{n\pi}\cos\frac{n\pi x}{6}\right]_0^6 + \left[\frac{36}{n^2\pi^2}\sin\frac{n\pi x}{6}\right]_0^6\right)$$

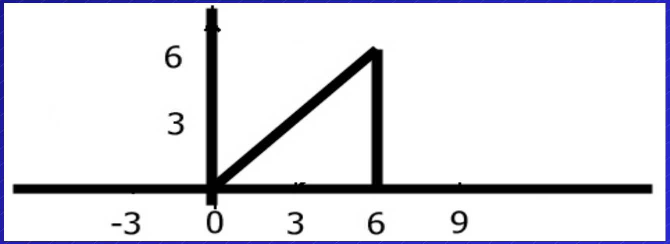
$$b_n = \frac{2}{d}\int_0^d f(x)\sin\frac{n\pi x}{d}\,dx = \frac{2}{6}\int_0^6 x\sin\frac{n\pi x}{6}\,dx$$

$$b_n = \frac{1}{3}\left(\left[-\frac{6x}{n\pi}\cos\frac{n\pi x}{6}\right]_0^6 + \int_0^6 \frac{6}{n\pi}\cos\frac{n\pi x}{6}\right) = \frac{1}{3}\left(\left[-\frac{6x}{n\pi}\cos\frac{n\pi x}{6}\right]_0^6 + \left[\frac{36}{n^2\pi^2}\sin\frac{n\pi x}{6}\right]_0^6\right)$$

$$b_n = \frac{1}{3}\left(\left[-\frac{36}{n\pi}\cos n\pi\right] + \left[\frac{36}{n^2\pi^2}\sin n\pi\right]\right) = \frac{12}{n^2\pi^2}\sin n\pi - \frac{12}{n\pi}\cos n\pi$$

n=1	n=2	n=3	n=4	n=5
$b_1 = \frac{12}{\pi}$	$b_2 = -\frac{12}{2\pi} = -\frac{6}{\pi}$	$b_3 = \frac{12}{3\pi} = \frac{4}{\pi}$	$b_4 = -\frac{12}{4\pi} = -\frac{3}{\pi}$	$b_5 = \frac{12}{5\pi}$

Find Half Range Sine Series which represents the displacement f(x), between x = 0 and 6, of the pulse *shown to the right*



Half range sine series

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{d}$$

where

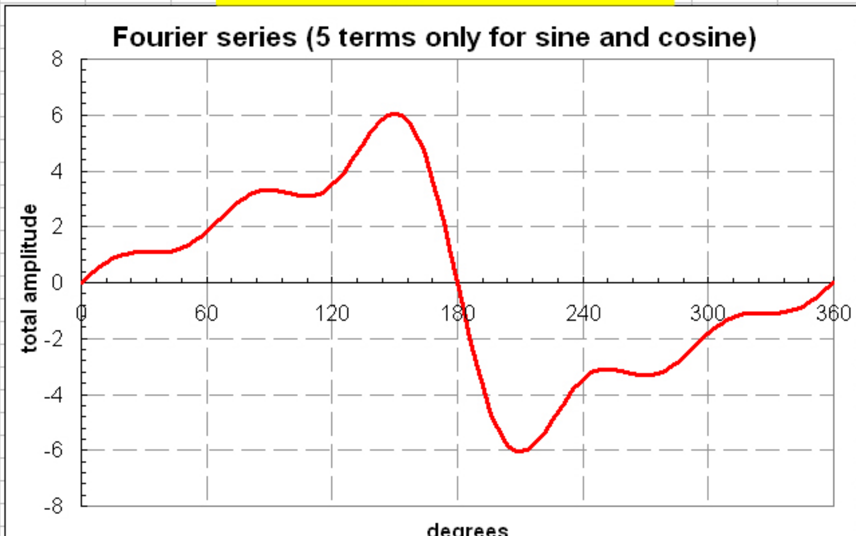
$$b_n=\frac{2}{d}\int_0^d f(x)\sin\frac{n\pi x}{d}\,dx$$

$$f(x)=\frac{12}{\pi}\sin\frac{\pi x}{6}-\frac{6}{\pi}\sin\frac{\pi x}{3}+\frac{4}{\pi}\sin\frac{\pi x}{2}-\frac{3}{\pi}\sin\frac{2\pi x}{3}+\frac{12}{5\pi}\sin\frac{5\pi x}{6}+.....$$

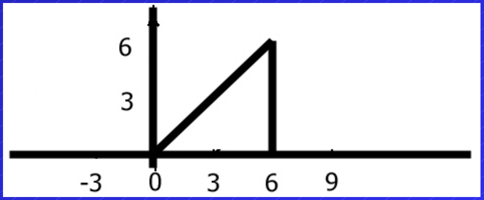
$$f(x) = \frac{12}{\pi}\sin\frac{\pi x}{6} - \frac{6}{\pi}\sin\frac{\pi x}{3} + \frac{4}{\pi}\sin\frac{\pi x}{2} - \frac{3}{\pi}\sin\frac{2\pi x}{3} + \frac{12}{5\pi}\sin\frac{5\pi x}{6} + ...$$

amplitudes of the cosine harmonics						amplitudes of the sine harmonics				
value of a0	value of a1	value of a2	value of a3	value of a4	value of a5	value of b1	value of b2	value of b3	value of b4	value of b5
0	0	0	0	0	0	3.821656	-1.91083	1.273885	-0.95541	0.7643312

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$



Check our Fourier series using Fourier_checker.xls

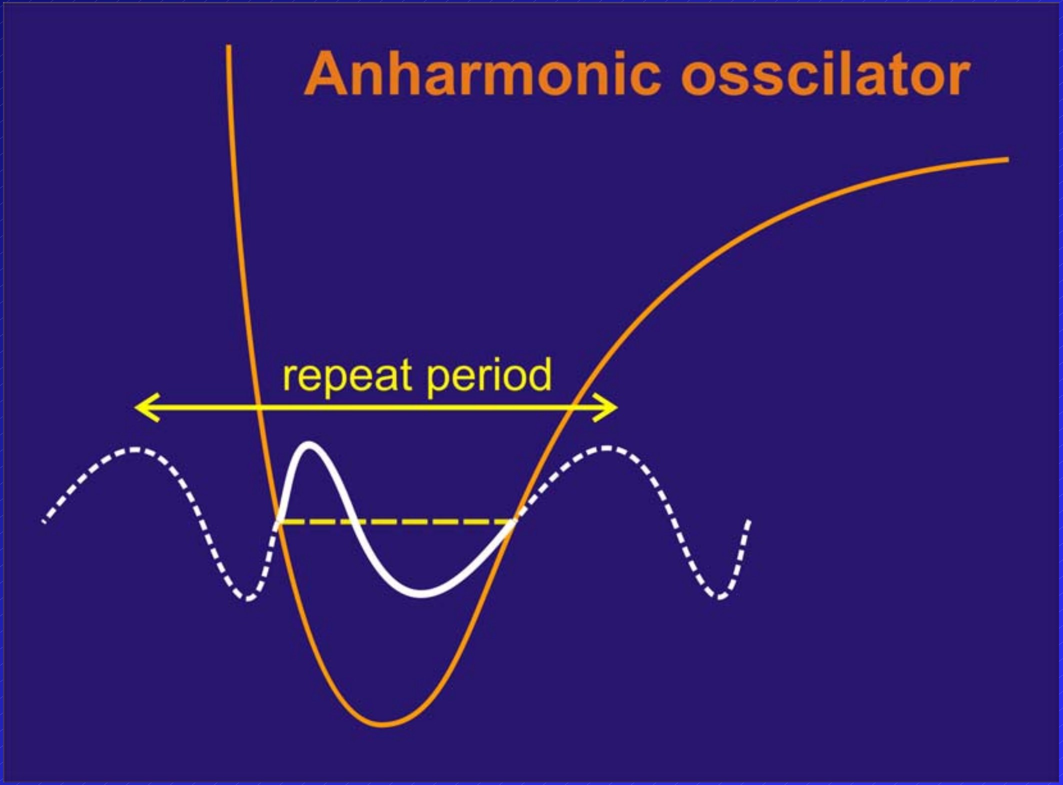


Fourier Series applied to pulses

Why is this useful?

In Quantum you have seen that there are specific solutions to the wave equation within a potential well subject to the given boundary conditions.

$$\Psi(x) = \sum_{n=1}^{\infty} \Psi_n(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{d}$$



Lecture 7 – Summary

- Practice questions – online at
http://www.hep.shef.ac.uk/phy226/unit1/phy226T1to4.htm

- Normal series
- Even and odd functions
- Pulses

Lecture 8 – Summary

- Practice questions – online at http://www.hep.shef.ac.uk/phy226/unit1/phy226T1to4.htm

- Complex Fourier series
- Parseval's theorum
- Revision & Practice

Lecture 8 – Fourier

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L}$$

$$a_0 = \frac{2}{L}\int_0^L f(x)dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}dx \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}dx$$

Complex Fourier Series

In many areas of physics, especially Quantum mechanics, it is more convenient to consider waves written in their complex form

$$\cos\frac{2n\pi x}{L} = \cos nx = \frac{1}{2}(e^{inx} + e^{-inx})$$

$$\sin\frac{2n\pi x}{L} = \sin nx = \frac{1}{2i}(e^{inx} - e^{-inx})$$

The complex form of the Fourier series can be derived by assuming a solution of the form $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ and then multiplying both sides by e^{-imx} and integrating over a period

Complex Fourier Series

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$$\int_0^{2\pi} f(x) e^{-imx} dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{inx} e^{-imx} dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{i(n-m)x} dx$$

Complex Fourier Series

$$\int_0^{2\pi} f(x)e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{inx}e^{-imx}dx = \sum_{n=-\infty}^{\infty} c_n \int_0^{2\pi} e^{i(n-m)x}dx$$

$$\int_0^{2\pi} e^{i(n-m)x}dx = \int_0^{2\pi} \cos(n-m)x dx + i\int_0^{2\pi} \sin(n-m)x dx$$

For $n \neq m$ integral vanishes. For *n=m* integral = 2π. So $c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx$

For a period of 2π the complex Fourier series is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\, dx$$

The more general expression with **period *L*** is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2\pi inx}{L}}$$

where

$$c_n = \frac{1}{L}\int_0^{L} f(x) e^{-\frac{2\pi inx}{L}}\, dx$$

Example 4.5

Find the complex Fourier series for $f(x) = x$ in the range $-2 < x < 2$ if the repeat period is 4.

$$c_n = \frac{1}{L}\int_0^L f(x) e^{\frac{-2\pi inx}{L}}\, dx$$

and period is 4, so we write

$$c_n = \frac{1}{4}\int_{-2}^{2} x e^{\frac{-\pi inx}{2}}\, dx$$

Integration by parts $\int u\, dv = uv - \int v\, du$

$$u = x \quad and \quad dv = e^{\frac{-\pi inx}{2}}\, dx$$

$$du = dx \;\; and \;\; v = \frac{-2}{\pi in} e^{\frac{-\pi inx}{2}}$$

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$$du = dx \;\; and \;\; v = \frac{-2}{\pi i n} e^{\frac{-\pi i n x}{2}}$$

$$c_n = \frac{1}{4}\left[\frac{-2x}{\pi i n} e^{\frac{-\pi i n x}{2}} + \int \frac{2}{\pi i n} e^{\frac{-\pi i n x}{2}}\,dx\right]_{-2}^{2} = \frac{1}{4}\left[\frac{-2x}{\pi i n} e^{\frac{-\pi i n x}{2}} - \frac{4}{\pi^2 i^2 n^2} e^{\frac{-\pi i n x}{2}}\right]_{-2}^{2} = \left[\frac{-x}{2\pi i n} e^{\frac{-\pi i n x}{2}} + \frac{1}{\pi^2 n^2} e^{\frac{-\pi i n x}{2}}\right]_{-2}^{2}$$

$$C_n = \left[\frac{-1}{\pi i n} e^{-\pi i n} + \frac{1}{\pi^2 n^2} e^{-\pi i n}\right] - \left[\frac{1}{\pi i n} e^{\pi i n} + \frac{1}{\pi^2 n^2} e^{\pi i n}\right] = \frac{-1}{\pi i n}\left(e^{-\pi i n} + e^{\pi i n}\right) + \frac{1}{\pi^2 n^2}\left(e^{-\pi i n} - e^{\pi i n}\right)$$

Since $\frac{-1}{i}\times\frac{i}{i}=i$ then $C_n=\frac{i}{\pi n}\left(e^{-\pi in}+e^{\pi in}\right)+\frac{1}{\pi^2 n^2}\left(e^{-\pi in}-e^{\pi in}\right)$

We want to find actual values for C_n so it would be helpful to convert expression for C_n into ***sine*** and ***cosine*** terms using the standard expressions:-

$$\cos n\pi=\frac{1}{2}\left(e^{-\pi in}+e^{\pi in}\right)$$

$$\sin n\pi=\frac{-1}{2i}\left(e^{-\pi in}-e^{\pi in}\right)$$

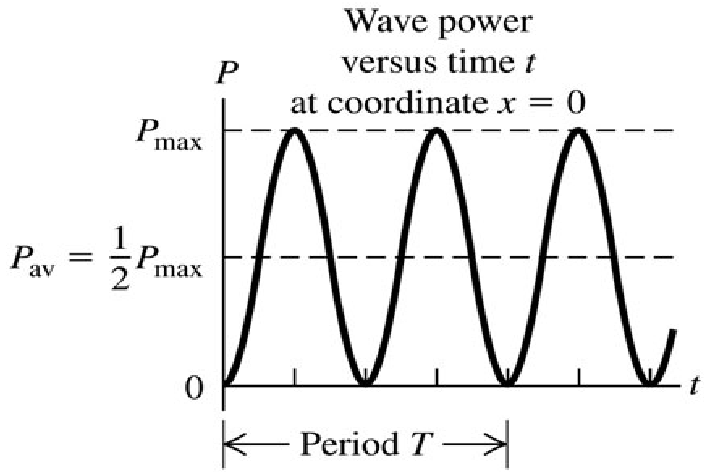
$C_n=\frac{2i}{\pi n}\cos n\pi-\frac{2i}{\pi^2 n^2}\sin n\pi=\frac{2i}{\pi n}\cos n\pi$ so $C_n=\frac{2i}{\pi n}\cos n\pi=\frac{2i}{\pi n}(-1)^n$ and since

$f(x)=\sum_{n=-\infty}^{\infty}c_n e^{\frac{2\pi inx}{L}}$ so $f(x)=\sum_{n=-\infty}^{\infty}\frac{2i}{\pi n}(-1)^n e^{\frac{\pi inx}{2}}$

Parseval's Theorem applied to Fourier Series

$$P_{max} = \sqrt{\mu T}\,\omega^2 A^2$$

$$P_{ave} = \frac{1}{2}\sqrt{\mu T}\,\omega^2 A^2$$



The energy in a vibrating string or an electrical signal is proportional to the square of the amplitude of the wave

NB, for all MECHANICAL waves, $P_{ave} \propto A^2\omega^2$

Parseval's Theorem applied to Fourier Series

Consider again the standard Fourier series with a period taken for

simplicity as 2π ….

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

Square both sides then integrate over a period:

$$\int_0^{2\pi} [f(x)]^2 dx = \int_0^{2\pi} \left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \right]^2 dx$$

$$\int_0^{2\pi} [f(x)]^2 dx = \int_0^{2\pi} \left[\frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \right]^2 dx$$

The RHS will give both squared terms and cross term.
When we integrate, *all* the cross terms will vanish.

All the squares of the cosines and sines integrate to give π (half the period).

$$\int_0^{2\pi} [f(x)]^2 dx = \pi \frac{a_0{}^2}{2} + \pi \sum_{n=1}^{\infty} [a_n{}^2 + b_n{}^2]$$

Parseval's theorem says the total energy in a vibrating system is equal to the sum of the energies in the individual modes.

Practice and revision

Try online questions 2 & 7