Topic 2

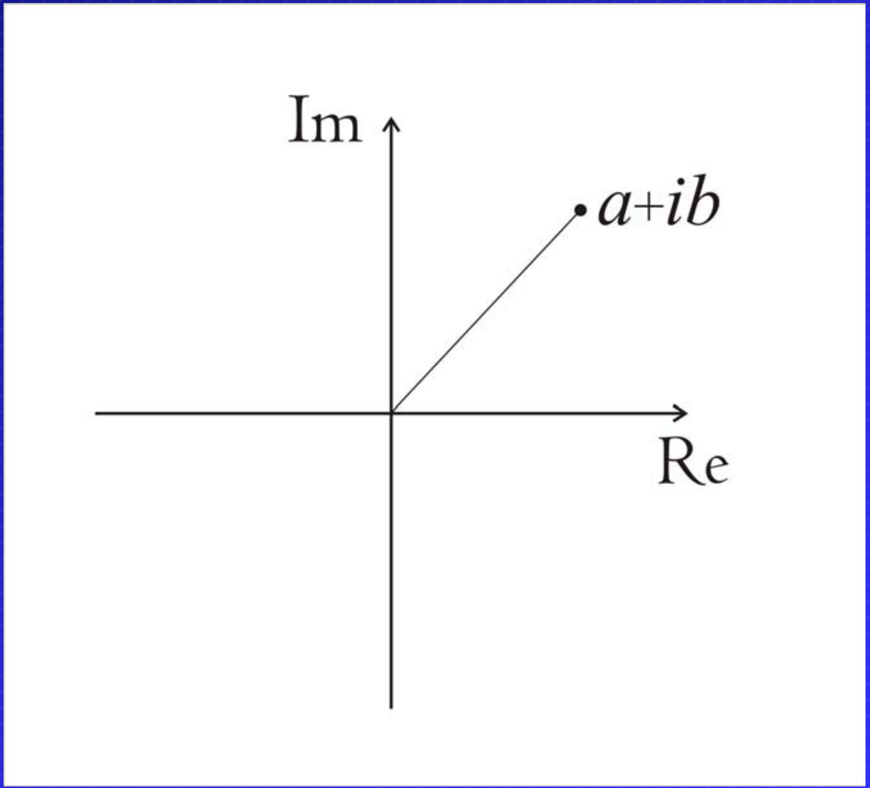
Complex numbers

Purpose of lecture

- Remind you of Cartesian and polar forms
- Conversion between them
- Powers and roots of complex numbers
- Integration by parts

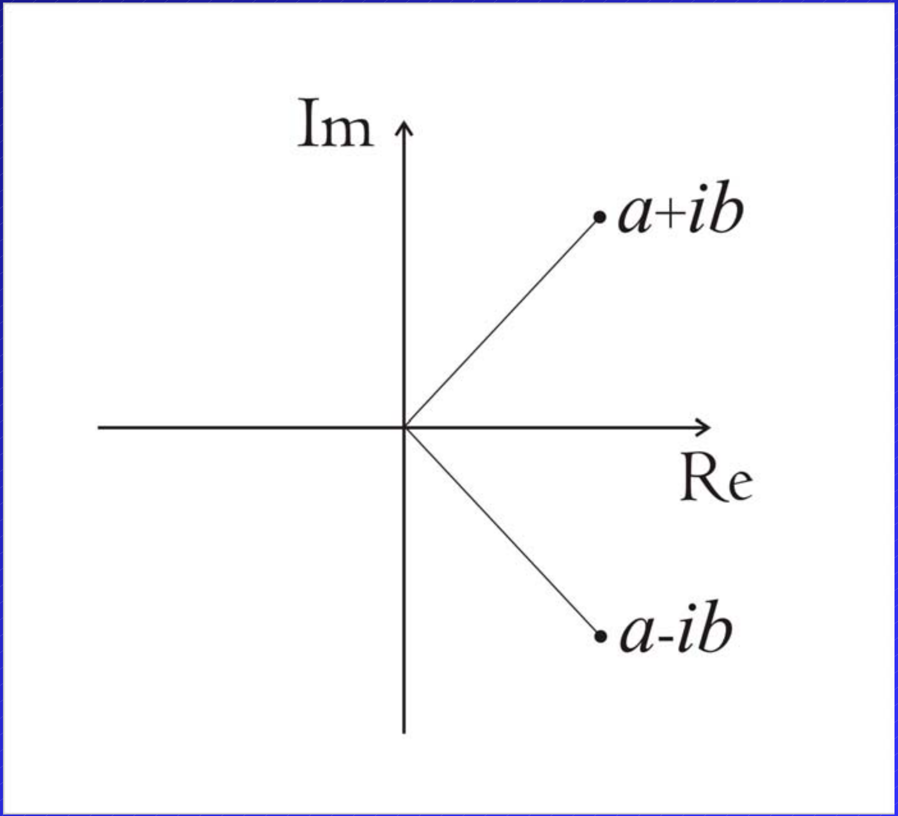
The Argand diagram

$$z = a + ib$$



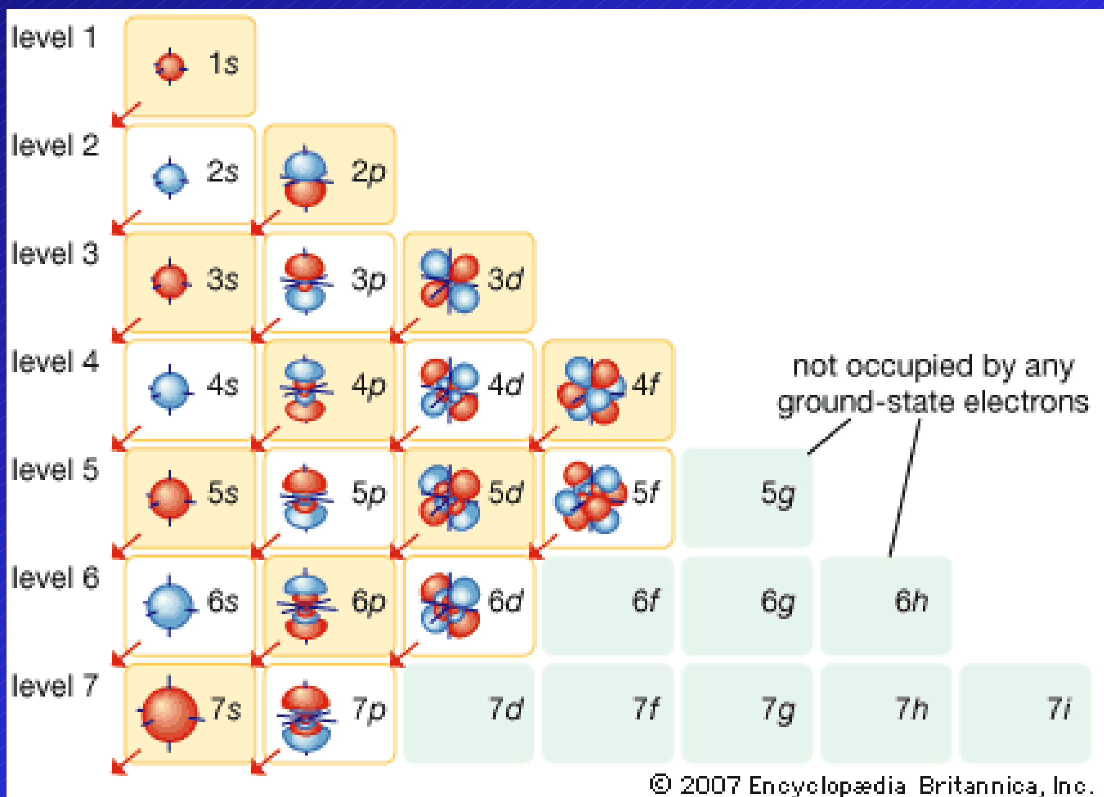
Complex conjugate

$$z^* = a - ib$$



Real answers from imaginary numbers

$$i\hbar\frac{\partial}{\partial t}\psi(r,t) = -\frac{\hbar^2}{2m}\nabla^2(r,t) + V(r)\psi(r,t)$$



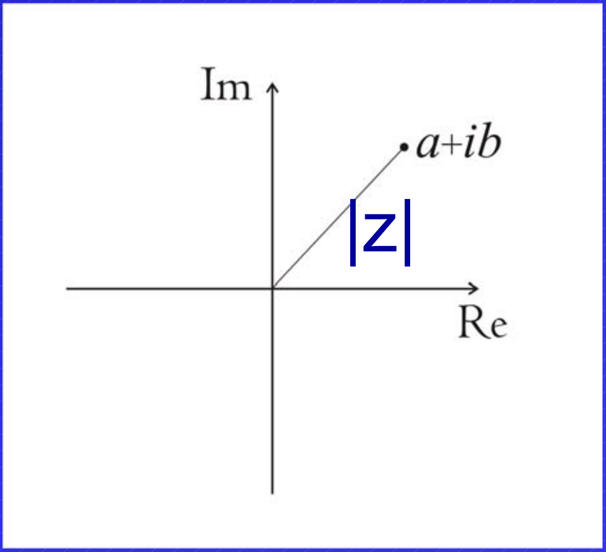
Why complex conjugate?

If $z = a + ib$ Evaluate

z^2

zz^*

How do we find $|z|$?



Why complex conjugate?

What is |z|?

What works - z^2 or zz^*?

$$z^2 = (a+ib)^2 = a^2 - b^2 + 2iab$$

$$zz^* = (a+ib)(a-ib) = a^2 + b^2$$

Real answers from imaginary numbers

$$|z| = \sqrt{zz^*}$$

$$|z| = \sqrt{a^2 + b^2}$$

Example 2.1

Find the modulus of

$$|3+4i|$$

Example 2.1 (ans)

$$|3+4i| = \sqrt{(3+4i)(3-4i)}$$

$$|3+4i| = \sqrt{9+16} = 5$$

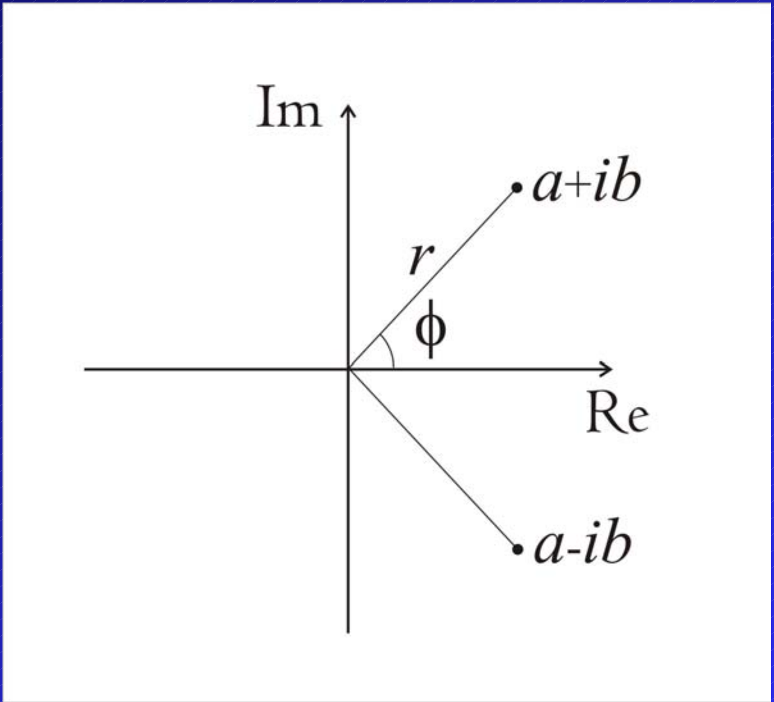
Eulers formula

$$e^{ix} = \cos x + i \sin x$$

$$re^{i\phi} = r\cos\phi + ri\sin\phi$$

Polar form

$$z = a + ib = re^{i\phi}$$



$$a = r\cos\phi$$

$$b = r\sin\phi$$

$$\phi = \tan^{-1}\left(\frac{b}{a}\right)$$

$$r = \sqrt{a^2 + b^2}$$

Why both forms?

Addition:

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2)$$

$$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$$

Subtraction:

$$z_1 - z_2 = (a_1 + ib_1) - (a_2 + ib_2)$$

$$z_1 - z_2 = (a_1 - a_2) + i(b_1 - b_2)$$

Why both forms?

Multiplication?

Division?

Why both forms?

Multiplication & division

$$z_1 z_2 = r_1 r_2 e^{i(\phi_1 + \phi_2)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)}$$

Example 2.2

Express in polar form:

$$\frac{1+i}{1+1.73i}$$

$$\tan^{-1}(1.73) = \pi / 3$$

$$\tan^{-1}(1) = \pi / 4$$

Example 2.2 (ans)

$$z_1 = 1 + i = \sqrt{2}\, e^{i\pi/4}$$

$$\tan\theta = \frac{1}{1} \qquad r = \sqrt{2}$$

$$\theta = \pi/4$$

$$z_2 = 1 + 1.73i$$

$$\tan\theta = 1.73 \qquad r = \sqrt{1+3} = 2$$

$$\theta = \pi/3$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}\, e^{i\pi/4}}{2 e^{i\pi/3}} = \frac{\sqrt{2}}{2} e^{i(\pi/4 - \pi/3)}$$

$$\frac{z_1}{z_2} = \frac{\sqrt{2}}{2} e^{-i\pi/12} = 0.707 e^{-i\pi/12}$$

Powers and roots

Powers are easy

$$z^n = r^n e^{in\phi}$$

Roots are trickier

$$z^{1/n} = r^{1/n} e^{i(\phi+2\pi p)/n}$$

Multiple roots

What's?

$$\sqrt{1}$$

$$\sqrt[3]{1}$$

$$\sqrt[n]{1}$$

Multiple roots

$$\sqrt{1} = +1, -1$$

Two roots

$$\sqrt[3]{1} = \begin{cases} 1 \\ -\frac{1}{2} + \frac{\sqrt{3}}{2} i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2} i \end{cases}$$

Three roots

Working….

Multiple roots

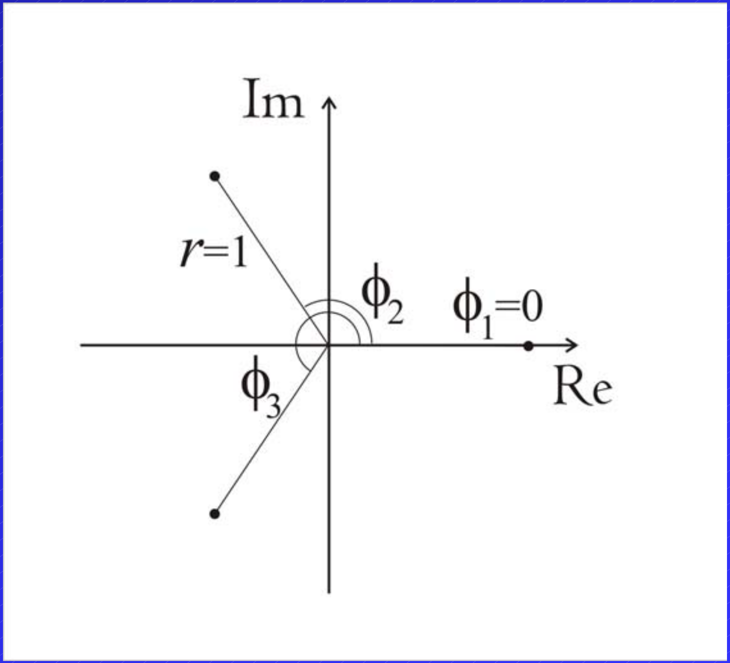
$$z^{1/n} = r^{1/n} e^{i(\phi + 2\pi p)/n}$$

$$1 = 1 + i0$$

$$r = 1 \text{ and } \phi = \tan^{-1}(0/1) = 0$$

$$z = e^{i(0+2\pi p)/3} \quad \text{where } p = 0,1,2$$

$$z = 1, e^{\frac{2i\pi}{3}}, e^{\frac{4i\pi}{3}}$$



Multiple roots

$\sqrt[n]{1} = 1^{1/n}$ Has n roots

$z^{1/n}$ Has n roots

n roots

Roots are easy $z^{1/n} = r^{1/n} e^{i(\phi+2\pi p)/n}$

Step 1: how many roots will there be?

Step 2: how many values of p?

Step 3: work them out one at a time…

Example 2.3 (Q)

If $z = 9e^{i\pi/3}$ find $z^{1/2}$

Example 2.3 (A)

Step 1: write down z in polars with the $2\pi p$ bit added on to the argument. $z = 9e^{i(\pi/3 + 2\pi p)}$

Step 2: say how many values of p you'll need and write out the rooted expression here n = 2 so I'll need 2 values of p; p = 0 and p = 1 $z^{½} = \sqrt{9}\, e^{i(\pi/3 + 2\pi p)/2}$

Step 3: Work it out for each value of p: $z^{½} = 3e^{i(\pi/3)/2} = 3e^{i(\pi/6)}$ for p = 0, $z^{½} = 3e^{i(\pi/3 + 2\pi)/2} = 3e^{i(\pi/6 + \pi)}$ for p = 1

Remember that $e^{i\phi} = (\cos\phi + i\sin\phi)$ so $e^{i\pi} = -1$

Therefore we could write
$z^{½} = 3e^{i(\pi/6 + \pi)} = 3e^{i\pi/6}(e^{i\pi}) = -3e^{i\pi/6}$ for p =1, and $3e^{i(\pi/6)}$ for p = 0

Example 2.4 (Q)

If $z = 27e^{i\pi/2}$ find $z^{1/3}$

Example 2.4 (A)

Step 1 : $z = 27e^{i(\pi/2 + 2\pi p)}$

Step 2 : for $n=3$ $\quad p = 0, 1, 2$

Step 3 : for $p=0$ $\quad z^{1/3} = 3e^{i(\pi/2)/3} = 3e^{i\pi/6}$

for $p=1$ $\quad z^{1/3} = 3e^{i(\pi/2 + 2\pi)/3} = 3e^{i(5\pi/6)}$

for $p=2$ $\quad z^{1/3} = 3e^{i(\pi/2 + 4\pi)/3} = 3e^{i(9\pi/6)}$

So the 3 roots of $z^{1/3}$ are : $3e^{i\pi/6}, 3e^{i5\pi/6}, 3e^{i9\pi/6}$

Complex trig functions

Remember

$$e^{ikx} = \cos kx + i \sin kx$$

$$e^{-ikx} = \cos kx - i \sin kx$$

Complex trig functions

Remember

$$e^{ikx} = \cos kx + i \sin kx$$

$$e^{-ikx} = \cos kx - i \sin kx$$

So....

$$\cos kx = \tfrac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$$

$$\sin kx = \tfrac{1}{2i}\left(e^{ikx} - e^{-ikx}\right)$$

Complex differentiation

Remember that i is just a number!

$$\frac{d(e^{ikx})}{dx} = ike^{ikx}$$

Much easier than sine's and cosine's

Integration by parts - Remember the chain rule

$$\frac{d(f(x)g(x))}{dx} = f(x)\frac{d(g(x))}{dx} + \frac{d(f(x))}{dx}g(x)$$

$$f(x)g(x) = \int f(x)\frac{d(g(x))}{dx}dx + \int \frac{d(f(x))}{dx}g(x)dx$$

$$\int f(x)\frac{d(g(x))}{dx}dx = f(x)g(x) - \int g(x)\frac{d(f(x))}{dx}dx$$

$$v = g(x)$$

$$\frac{dv}{dx} = \frac{d(g(x))}{dx}$$

$$v = \int \frac{d(g(x))}{dx}dx$$

$$u = f(x)$$

$$\frac{du}{dx} = \frac{d(f(x))}{dx}$$

$$du = \frac{d(f(x))}{dx}dx$$

$$\int u\,dv = uv - \int v\,du$$

Integration by parts - example

$$\int u\,dv = uv - \int v\,du$$

evaluate $\int x\cos x\,dx$

Substitutions

$$u = x$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$dv = \cos x\,dx$$

$$v = \int \cos x\,dx$$

$$v = \sin x$$

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx$$

$$\int x\cos x\,dx = x\sin x + \cos x + c$$

Integration by parts - example

$$\int u\,dv = uv - \int v\,du$$

evaluate $$\int xe^x dx$$

Substitutions

$$u = x$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$dv = e^x dx$$

$$v = \int e^x dx$$

$$v = e^x$$

$$\int xe^x dx = xe^x - \int e^x dx$$

$$\int xe^x dx = e^x(x-1) + c$$

Integration by parts - example

$$\int u\,dv = uv - \int v\,du$$

evaluate

$$\int \ln x\,dx$$

Substitutions

$$u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$du = \frac{dx}{x}$$

$$dv = 1dx$$

$$v = x$$

$$\int \ln x\,dx = x\ln x - \int \frac{x}{x}\,dx$$

$$\int \ln x\,dx = x(\ln x - 1) + c$$

Complex numbers

Summary

- Remind you of Cartesian and polar forms
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