THE UNIVERSITY OF SHEFFIELD

DEPARTMENT OF PHYSICS AND ASTRONOMY

Autumn Semester 2004-2005 2 hours

MATHEMATICAL METHODS FOR PHYSICS AND ASTRONOMY

Answer question ONE (compulsory) and TWO others

A formula sheet and table of physical constants is attached to this paper.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

COMPULSORY

1 (a) Put the following numbers into the form $re^{i\theta}$ and also $x + iy$:

$\sqrt{i}$ and $\frac{1}{(1+i)^2}$. [1]

(b) Write $\frac{2}{1-x^2}$ in partial fractions. [1]

(c) Expand $(1-x)^{3/2}$ in powers of x up to and including the term in x^2. Show that the expansion is valid to ~0.5% for $x = 0.1$. [1]

(d) Solve the equation $\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \alpha^2 x$ for $t \geq 0$ subject to the boundary conditions that $\int_0^\infty \mathrm{d}t\ x(t)$ is finite and $x = 1$ at $t = 0$. [1]

(e) Assuming that $\frac{1}{\pi}\int_0^{2\pi} \mathrm{d}\theta \sin n\theta \cos m\theta = 0$, where n and m are integers, evaluate

$$\frac{2}{L}\int_0^L \mathrm{d}x \sin\frac{2n\pi x}{L}\cos\frac{2m\pi x}{L}.$$ [2]

(f) By defining $\varphi = \theta - \pi$, or otherwise, show that

$$\int_{-\pi}^{\pi} \mathrm{d}\varphi \sin n\varphi \cos m\varphi = 0.$$ [1]

(g) An odd function $g(\theta)$ satisfies $g(\theta) = -g(-\theta)$. It also satisfies $g(\theta + 2\pi) = g(\theta)$.
Show that this can be expanded in a *half range* sine series,

$$g(\theta) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

Write down an expression for the coefficients, b_n. [2]

(h) In a counting experiment there is an average of one count per second. What is the probability that in one second there are
(i) exactly one count and
(ii) exactly five counts? [1]

[You may assume the Poisson distribution,

$P_n = \frac{\mu^n}{n!}e^{-\mu}$; $\mu \equiv \langle n \rangle$ applies to this experiment].

2 A displacement, $y(x,t)$ that satisfies the wave equation

$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

is defined in region $0 < x < L$. The boundary conditions are

$$\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial y(x,t)}{\partial x}\right|_{x=L} = 0 \text{ for all times } t.$$

(a) Explain why it is appropriate to solve this problem using separation of the variables, $y(x,t) = X(x)T(t)$. [1]

(b) Show that $\frac{\mathrm{d}^2 X(x)}{\mathrm{d}x^2} = AX(x)$. [2]

(c) Find values of A that are consistent with the boundary conditions. [2]

(d) Find the frequencies of the normal modes. [2]

(e) At $t = 0$ the velocity is zero and the displacement is given by

$$y(x,0) = d\left[\sin\frac{2\pi x}{L} + \frac{1}{3}\sin\frac{3\pi x}{L}\right],$$

where d is a constant.

Calculate the displacement at all subsequent times, $y(x,t)$. [2]

(f) Sketch the displacement of the centre of the string, $y(L/2, t)$, as a function of time. [1]

3 (a) Give an example of the use of Fourier Transforms in physics or astronomy. [1]

(b) Show that the following Fourier Transform may be written as a cosine transform as shown:

$$F_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; e^{-\alpha|x|} e^{ikx} = \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} \mathrm{d}x \; e^{-\alpha x} \cos kx \, . \quad [3]$$

(c) Show that $F_1(k) = \dfrac{2a}{\sqrt{2\pi}(a^2 + k^2)}$. [3]

(d) Evaluate the three dimensional Fourier Transform of the function $f(r) = \dfrac{e^{-\alpha r}}{r}$ given by

$$F_2(k) = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} \mathrm{d}\phi \int_0^{\infty} r e^{-\alpha r} \mathrm{d}r \int_0^{\pi} \mathrm{d}\theta \sin\theta \, e^{ikr\cos\theta}$$

Explain why the value of the function $F_2(k)$ is small for $k >> \alpha$. [3]

4(a) (i) Show that the wave equation in three dimensions,

$$\frac{\partial^2\Psi(x,y,z,t)}{\partial x^2}+\frac{\partial^2\Psi(x,y,z,t)}{\partial y^2}+\frac{\partial^2\Psi(x,y,z,t)}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2\Psi(x,y,z,t)}{\partial t^2},$$

where c is the wave velocity, can be solved by separation of the variables. [2]

(ii) Hence show that a possible solution is a plane wave,
$\Psi(x,y,z,t)=A\exp(i\omega t-ik_x x-ik_y y-ik_z z)=A\exp i\left(\omega t-\mathbf{k.r}\right),$
where $k_x^2+k_y^2+k_z^2=k^2=\dfrac{\omega^2}{c^2}$. [1]

(iii) What is the direction of propagation of this wave? [1]

(iv) How is the wavelength related to the angular frequency ω? [1]

(b) (i) Show that a spherical wave,

$$\Psi(r,t)=\frac{A\exp\left(\pm ikr+i\omega t\right)}{r},$$

is the solution to the wave equation in spherical coordinates (assuming a spherical wave amplitude),

$$\nabla^2\Psi(r,t)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\Psi(r,t)\right)=\frac{1}{c^2}\frac{\partial^2\Psi(r,t)}{\partial t^2}.$$ [3]

(ii) Explain the physical significance of the sign of $\pm ikr$ in the solution. What is the wavelength of the wave in this case? [1]

(iii) Show that, if $\left|\Psi(r,t)\right|^2$ can be taken to be proportional to the intensity of a source of radiation, then the inverse square law is obeyed by this source. [1]

5 (a) An unbiased coin is tossed 10 times. Find the probabilities that there are exactly 1, 3, 5, 7 heads. [4]

(b) The probability of finding a molecule moving with momentum p_x is given by

$$F(p_x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{p_x^2}{2\sigma^2}\right) \quad \text{where } \sigma^2 = 2mk_BT \text{ .}$$

(i) Show that this probability function is normalised. [1]

(ii) Find the values of $\langle p_x \rangle, \langle p_x^2 \rangle, \langle p_x^4 \rangle$. [3]

(iii) Show that Δp_x^2, defined by $\Delta p_x^2 = \left\langle \left(p_x^2 - \langle p_x^2 \rangle \right)^2 \right\rangle$, may be expressed as $\Delta p_x^2 = \langle p_x^4 \rangle - \langle p_x^2 \rangle^2$. [1]

Evaluate Δp_x^2 for this probability function. [1]

[In this question $\langle A \rangle$ means the average value of the quantity A.]

You may assume that $\int_{-\infty}^{\infty} x^4 e^{-ax^2} \mathrm{d}x = \frac{3}{4}\sqrt{\frac{\pi}{a^5}}$.

END OF QUESTION PAPER