THE UNIVERSITY OF SHEFFIELD

DEPARTMENT OF PHYSICS AND ASTRONOMY

Autumn Semester 2003 | 2 hours

MATHEMATICAL METHODS FOR PHYSICS AND ASTRONOMY

Answer question ONE (COMPULSORY) and TWO others.

A formula sheet and table of physical constants is attached to this paper.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

1 COMPULSORY

(a) Plot $e^{i\pi/3}$ and $\dfrac{1}{\sqrt{2}}(1-i)$ on an Argand diagram.
Evaluate $\sqrt{3+4i}$. [2]

(b) Which of the following equations have a solution that satisfies $\lim_{t\to\infty} f(t) \to 0$?

(i) $\dfrac{\mathrm{d}f(t)}{\mathrm{d}t} = -\lambda f(t)$;

(ii) $\dfrac{\mathrm{d}f(t)}{\mathrm{d}t} = af(t)$;

(iii) $\dfrac{\mathrm{d}f(t)}{\mathrm{d}t} = i\alpha f(t)$;

(iv) $\dfrac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} = -\omega_0^2 f(t)$;

(v) $\dfrac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} = b^2 f(t)$;

(vi) $\dfrac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2} = 2i\Lambda^2 f(t)$. [3]

(λ, a, α, ω_0, b, and Λ are positive constants)

(c) Figures A,B,C,D on page 3 each show one period of a periodic function. Figures a,b,c,d show the magnitude of the Fourier coefficients. Assign a Fourier spectrum to each function and give brief reasons for your choice. [4]

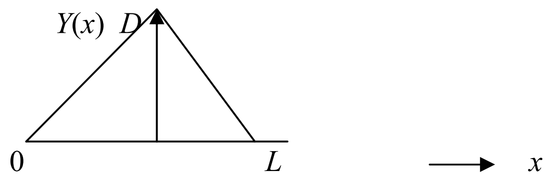
(d) Write down the element of volume integration using spherical polar co-ordinates and hence obtain the volume of a sphere of radius R by integration. [1]

A	**B**
C	**D**
a	**b**
c	**d**

2 Answer ***both*** parts of the question.

PART ONE

The allowed modes for a string originally in the shape shown are given below.



$$Y(x) = \frac{8D}{\pi^2}\left[\sin\frac{\pi x}{L} - \frac{1}{9}\sin\frac{3\pi x}{L} + \frac{1}{25}\sin\frac{5\pi x}{L} - \frac{1}{49}\sin\frac{7\pi x}{L} + \dots\dots\dots\right]$$

$$= \frac{8D}{\pi^2}\sum_{r=0}\frac{(-1)^r}{(2r+1)^2}\sin\frac{(2r+1)\pi x}{L}.$$

(a) Explain why this series can be identified as a 'half range Fourier sine series'. [1]

(b) Show that the series is periodic in $2L$ [*i.e.* it satisfies the condition that $Y(x)= Y(x+2L)$]. [1]

(c) Explain why the terms in $\sin\frac{2n\pi x}{L}$ where n is an integer, are missing. [1]

PART TWO

Consider a function defined for $0 < t < \tau$, $f(t) = \frac{t}{\tau}$. Sketch the function $f(t)$ for $-2\tau < t < 2\tau$, if extended as,

(d) a full range Fourier series; [1]

(e) a half range sine series; [1]

(f) a half range cosine series. [1]

Which of these series do you think might converge fastest and why? [1]

Evaluate the coefficients in the half range cosine series. [3]

3 (a) Show that the Fourier transform of a Gaussian

$$f(x) = \frac{1}{a\sqrt{2\pi}} \exp\left(-\frac{x^2}{2a^2} \right)$$

is also a Gaussian such as,

$$g(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x \; f(x) \exp(-ikx) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{a^2k^2}{2} \right).$$ [4]

Sketch both these functions and comment on their relative widths. [1]

(b) State the convolution theorem. [1]

Two Gaussians, $f(x) = A\exp\left(-\frac{x^2}{2a^2} \right)$ and $h(x) = B\exp\left(-\frac{x^2}{2b^2} \right)$,

are convolved together. Show that the result is a Gaussian with width $c = \sqrt{a^2 + b^2}$. [3]

Give one example of the use of the convolution theorem. [1]

4 (a) Explain why it is necessary to use spherical coordinates in order to solve problems of partial differential equations with spherical boundary conditions. [1]

Find the allowed value of k such that $\Psi(r,t) = \frac{Ae^{\pm ikr + i\omega t}}{r}$ is a spherical solution to the wave equation

$$\nabla^2 \Psi(r,t) = \frac{1}{c^2} \frac{\partial^2 \Psi(r,t)}{\partial t^2}.$$ [3]

What is the significance of $\pm$ in the exponential? [1]

(b) Show that the solution to Laplace's equation $\nabla^2 V(\mathbf{r}) = 0$ may be written as,

$$V(r,\theta,\phi) = \sum_{l,m} R_l(r) P_l^m(\theta) e^{im\phi}.$$

Here $R_l(r)$ and $P_l^m(\theta)$ satisfy the equations,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}R_l(r)}{\mathrm{d}r} \right) = l(l+1)R_l(r)$$

$$\frac{1}{\sin\theta} \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta \frac{\mathrm{d}P_l^m(\theta)}{\mathrm{d}\theta} \right) - \frac{m^2 P_l^m(\theta)}{\sin^2\theta} + l(l+1)P_l^m(\theta) = 0.$$ [4]

Explain why m needs to be an integer. [1]

5 Consider an experiment performed N times with an outcome x_i that occurs N_i times.

If $p_i = \dfrac{N_i}{N}$ and show that $\sum_i p_i = 1$. [1]

If the mean value is defined by $\langle x_i \rangle = \sum_i x_i p_i$ and the variance is defined by $\sigma^2 = \sum_i p_i \left(x_i - \langle x_i \rangle \right)^2$ show that,

$$\sigma^2 = \langle x^2 \rangle - \langle x \rangle^2 .$$ [2]

A particle is executing simple harmonic motion about its equilibrium position $x = 0$ so that $x(t) = a \cos \omega t$. The probability of finding the particle between x and $x + \mathrm{d}x$ is proportional to the time it spends in that interval $\dfrac{\mathrm{d}x}{(\mathrm{d}x/\mathrm{d}t)}$. [4]

Find the (normalised) probability function for the particle. [1]

Sketch this as a function of x.

Find the mean position of the particle and its standard deviation σ^2. [2]

END OF QUESTION PAPER