The University Of Sheffield.

DEPARTMENT OF PHYSICS & ASTRONOMY

Data Provided: *A formula sheet and table of physical constants is attached to this paper.*

Autumn Semester 2007-2008

MATHEMATICAL METHODS FOR PHYSICS & ASTRONOMY

2 HOURS

Answer Question ONE (COMPULSORY) and TWO additional questions.

All questions are marked out of ten. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

1. COMPULSORY

(a) Let $z=-1+\sqrt{3}i$. Plot z on an Argand diagram, express z in the form $z=re^{i\phi}$, and hence find $z^{1/2}$. [2]

(b) Write down the general solution of the equation of $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\lambda^2 x$. Find the particular solution such that at t = 0, x = x_0 and $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0}=0$. [2]

(c) Consider the integrals given below. For any which are identically equal to zero, explain briefly *why* they are zero. Evaluate any which are *not* zero.

$$\int_0^{2\pi}\cos 3x\sin x\,\mathrm{d}x$$

$$\int_0^{2\pi}\sin^2 3x\,\mathrm{d}x$$

$$\int_0^{2\pi}e^{3ix}\,\mathrm{d}x$$

[3]

(d) Evaluate the integral $\int_{-\infty}^{\infty}\delta(x-a)e^{ikx}\mathrm{d}x$ where $\delta(x-a)$ is a Dirac delta function. [1]

(e) In spherical polar coordinates (r, θ, ϕ), state the range of values each variable should take in order to cover all space. [1]

(f) In spherical polar coordinates the volume element is $\mathrm{d}V=r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$. By explicitly evaluating the integral, show that $\int_{\substack{\textit{sphere of}\\ \textit{radius } R}}\mathrm{d}V=\frac{4}{3}\pi R^3$. [1]

CONTINUED

2.

(a) An oscillator with natural frequency ω_0 is subject to damping with damping constant $2b$, so obeys the differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + 2b\frac{\mathrm{d}x}{\mathrm{d}t} + \omega_0^2 x = 0\,.$$

(i) For what value of b is the oscillator 'critically damped'? Find the general solution of the equation in this case. [2]

(ii) At $t = 0$, $x = 0$ and $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{t=0} = V_0$. Find the particular solution. [1]

(b) In a region of zero potential, the one dimensional time-dependent Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi\left(x,t\right)}{\partial x^2} = i\hbar\frac{\partial \Psi\left(x,t\right)}{\partial t}\,.$$

(i) Show that the equation has solutions of the form $\Psi(x,t) = u(x)T(t)$ such that $\frac{i\hbar}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = constant$, and find a corresponding ordinary differential equation for $u(x)$. [2]

(ii) Let the constant in part (i) be E, that is, let $\frac{i\hbar}{T}\frac{\mathrm{d}T}{\mathrm{d}t} = E$. Find the general solutions of the differential equations for $T(t)$ and $u(x)$. [2]

(iii) If $\Psi(0,t) = \Psi(L,t) = 0$ for all t, show that the general form of $\Psi(x,t)$ is $\Psi(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-iE_n t/\hbar}$, where n is an integer, and give an appropriate expression for E_n. [3]

3.

The Fourier Series of a function $f(x)$ of period 2π is

$$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\ a_n\cos nx+b_n\sin nx\ .$$

(a) Write down expressions for the coefficients a_0, a_n and b_n. [2]

A certain function $f(x)$ has period 2π and on the range $-\pi < x \le \pi$ is defined as

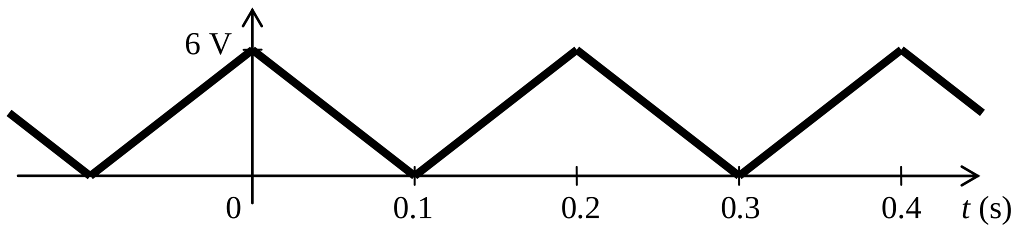
$$f(x)=\begin{cases}\dfrac{\pi}{2}+x, & -\pi<x\le 0\\[2ex] \dfrac{\pi}{2}-x, & 0<x\le\pi\end{cases}\quad .$$

(b) Sketch the function over the range $-2\pi \le x \le 2\pi$. [2]

(c) Explain why we can expect the Fourier series of this function to contain no constant term and no sine terms. [1]

(d) Show that the function has Fourier series $f(x)=\dfrac{4}{\pi}\displaystyle\sum_{n\ odd}\frac{1}{n^2}\cos nx$. [3]

(e) From your answer to part (d) or otherwise, find a Fourier series for the voltage signal shown below: [2]



4.

The Fourier transform of a function $f(x)$ is defined as

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} \mathrm{d}x \,.$$

(a) Show that if $f(x)$ is even, the formula can be written as

$$F(k) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos kx \, \mathrm{d}x \,.$$ [1]

(b) In a similar way, find an expression for $F(k)$ valid only when $f(x)$ is odd. [1]

(c) If $f(x)$ is real, under what conditions is its Fourier transform real? [1]

(d) Find the Fourier transform of the function $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| \geq a \end{cases}$. [1]

(e) For the function in part (d), sketch the function and its transform. [2]

(f) Show that the Fourier transform of the function $f(x) = \begin{cases} \cos x, & |x| < \pi/2 \\ 0, & |x| \geq \pi/2 \end{cases}$

is $F(k) = \sqrt{\frac{2}{\pi}} \frac{\cos k\pi/2}{1 - k^2}$. [3]

(g) Briefly describe one situation in physics or astronomy in which Fourier transforms are encountered or used. [1]

5.

(a) In two dimensional plane polar coordinates (r, θ), Laplace's equation for a potential $U(r, \theta)$ is

$$\nabla^2 U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} = 0 .$$

(i) Show by substitution that $U_1 = r\cos\theta$ is a solution of this equation. [1]

(ii) Show similarly that $U_n = r^n \cos n\theta$ is a solution for any integer n. [2]

(b) In three dimensional spherical polar coordinates (r, θ, ϕ) and when the potential $V(r)$ is a function of r only, Laplace's equation becomes

$$\nabla^2 V(r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}V(r)}{\mathrm{d}r}\right) = 0 .$$

Show, by directly solving the equation, that the only solutions are of the form $V(r) = \dfrac{A}{r} + B$ where A, B are constants. [3]

(c) Find the relationship between k and ω such that the spherical wave

$$\Psi(r,t) = \frac{A}{r} e^{i(kr - \omega t)}$$

is a solution of the wave equation $\nabla^2 \Psi = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \Psi}{\partial r}\right) = \dfrac{1}{c^2}\dfrac{\partial \Psi}{\partial t^2}$. [3]

(d) Explain why many three-dimensional problems in physics and astronomy are more easily solved using spherical polar coordinates than Cartesian coordinates. [1]

END OF QUESTION PAPER