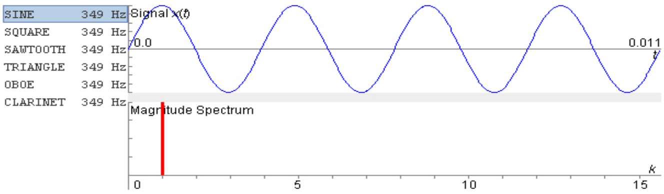
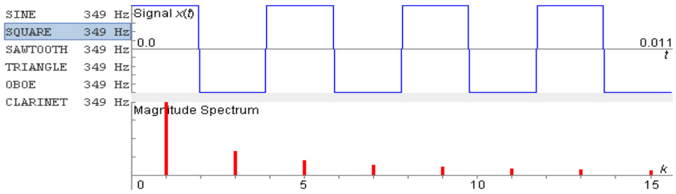
Lectures 9-11: Fourier Transforms

References *Jordan & Smith* Ch.27, *Boas* Ch.15 section 4, *Kreyszig* Ch.11.
Web site http://www.jhu.edu/signals/: go to Continuous Time Fourier Transform Properties.

1. Introduction to Fourier Transforms

We have seen that any periodic function can be represented by a Fourier series and that any pulse can be represented by a half range Fourier series. In both cases the shape is formed from summed sine and/or cosine terms of specific harmonic frequencies and amplitudes. Sometimes we may wish to study this distribution of frequencies rather than stare at the final summed Fourier series. See below from jhu.edu.





Two questions may occur to us:

1) Is there anything analogous to a Fourier series for a function which is *not* periodic?
2) The frequencies included in the Fourier sum is an infinite set (*n* takes values from 0 to infinity) but by no means includes all possible frequencies – it is a discrete set of frequencies (*n* is restricted to integer values), not a continuous spectrum. Can we somehow modify the series to contain a continuous spectrum?

Since an integral is the limit of a sum, you may not be surprised to learn that in the above cases the Fourier *series* (sum) is replaced by the Fourier *transform* which describes the frequencies present in the original function .

Fourier transforms, sometimes called Fourier integrals can be used to represent

- non-periodic functions, e.g. a single voltage pulse, a flash of light
- a continuous spectrum of frequencies, e.g. a continuous range of colours of light or musical pitch

They are used extensively in all areas of physics and astronomy.

2. Definition of Fourier Transforms

In our study of Fourier series we focussed mainly on the sine and cosine forms but with Fourier transforms it is usually more convenient to use a complex exponential form. The formulae are:

$$f(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\,d\omega \quad \text{where} \quad F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$$

Similarly,

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk \quad \text{where} \quad F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

Essentially the formula for *F(k)* defines the Fourier transform of *f(x)*; later we will prove that substituting this into the other formula returns the original function *f(x)*. First we will assume the formulae are true, learn to use them and observe some important properties.

The functions *f(x)* and *F(k)* (similarly *f(t)* and *F(ω)*) are called a pair of **Fourier transforms or integrals**. The only difference in the form of the integrals is the sign of the exponent, and in practice it is common to call either function the Fourier transform of the other.

Notes

1. *t* and ω, or *x* and *k*, are often called *conjugate variables.*
2. *k* is the *wavenumber,* $k = 2\pi/\lambda$ (compare with $\omega = 2\pi/T$).
3. There are different conventions about the factor of 2π. The convention we are using, with $1/\sqrt{2\pi}$ appearing symmetrically, is the most commonly used by physicists.

Applications

Fourier transforms are used in many diverse areas of physics and astronomy. For example:

(i) In the optics course you will find that the intensity of the Fraunhofer diffraction pattern from an aperture is the modulus squared of the Fourier transform of the aperture.

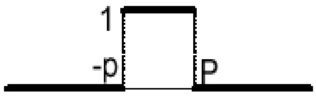
(ii) Next year in nuclear physics you will find that *any* (weak) scattering is found from the Fourier transform of the scattering potential. (Although nuclear physicists call it the 'form factor' instead of the Fourier transform.)

(iii) In quantum mechanics a localised particle has a spread of momenta. These are given by the Fourier transform of the wave packet.

3. Examples

Example 1: A rectangular ('top hat') function

Find the Fourier transform of the function $f(x) = \begin{cases} 1 & -p < x < p \\ 0 & -p > x \text{ and } x > p \end{cases}$

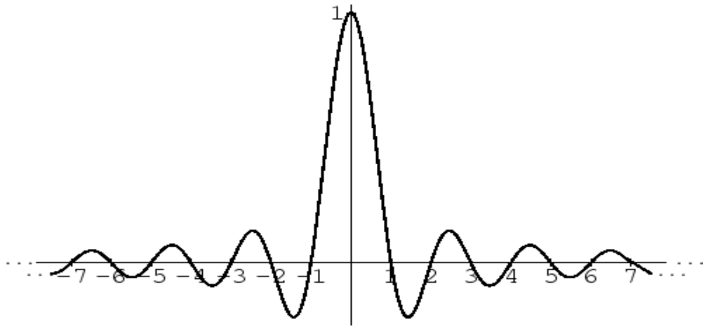


$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int_{-p}^{p} e^{-ikx}dx = \frac{-1}{\sqrt{2\pi}}\frac{1}{ik}\left[e^{-ikp} - e^{ikp}\right] = \frac{2p}{\sqrt{2\pi}}\frac{\sin kp}{kp}$$

This function occurs so often it has a name: it is called a sinc function.

$$\boxed{\text{sinc } x = \frac{\sin \pi x}{\pi x}}$$

(Or sometimes defined as $\frac{\sin x}{x}$)



Finding the value at $x = 0$ is a little tricky. The easiest method is to use the series expansion of sine and look at the limit as $x \to \infty$:

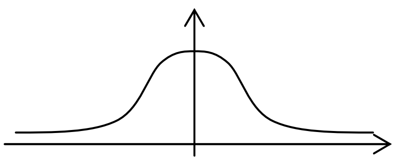
$$\sin \pi x = \pi x - \frac{(\pi x)^3}{3!} + .. = \pi x[1 - \frac{(\pi x)^2}{6} + ..]. \text{ Hence} \lim_{x\to 0}\left[\frac{\sin \pi x}{\pi x}\right] = \lim_{x\to 0}\left[\frac{\pi x\{1 - \frac{(\pi x)^2}{6}\}}{\pi x}\right] = \lim_{x\to 0}\{1 - \frac{(\pi x)^2}{6}\} = 1.$$

We see that the majority of the function lies inside the region $-1 < x < 1$. This is even more true of the function $\text{sinc}^2 x$, which is of interest in optics (see later).

The total area under the curve can be evaluated: $\int_{-\infty}^{\infty} dx \frac{\sin \pi x}{\pi x} = 1$

Example 2: The Gaussian

Find the Fourier transform of the Gaussian function $f(x) = \sqrt{\frac{a}{2\pi}}e^{-ax^2/2}$.



<u>*Note*</u>

The *Gaussian* is a function encountered frequently in Quantum Mechanics and statistics.

The constant a is related to the width: $f(x)$ falls to $1/e$ of its initial value at $x^2 = 2/a$, $x = \pm\sqrt{2/a}$.

The factor of $\sqrt{a/2\pi}$ ensures that $\int_{-\infty}^{\infty} f(x)dx = 1$ (as required for a probability distribution).

Using the formula above, $F(k) = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{a}{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-ax^2}{2}} e^{-ikx}dx \quad = \frac{\sqrt{a}}{2\pi}\int_{-\infty}^{\infty} e^{\frac{-ax^2}{2} - ikx}dx$

This integral is pretty tricky. It can be shown that $\int_{-\infty}^{\infty} e^{-nx^2 - jx} dx = \sqrt{\frac{\pi}{n}} e^{\frac{j^2}{4n}}$. Here $n = \frac{a}{2}$ and $j = ik$.

So $F(k) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} e^{\frac{-ax^2}{2} - ikx} dx = \frac{\sqrt{a}}{2\pi} \left[\sqrt{\frac{2\pi}{a}}\ e^{\frac{-k^2}{2a}} \right] = \frac{1}{\sqrt{2\pi}}\ e^{\frac{-k^2}{2a}}$

Hence we have found that the Fourier transform of a Gaussian is a Gaussian!
(The inverse transform can be performed in a very similar way – of course giving the initial Gaussian.

Relevance to Quantum mechanics

Let us look at the widths of the Gaussian $f(x) = \sqrt{\frac{a}{2\pi}} e^{-ax^2/2}$ and its transform $F(k) = \frac{1}{\sqrt{2\pi}}\ e^{\frac{-k^2}{2a}}$.

We could take any width we wanted (such as full width half maximum, the mean squared value of x, or even sigma) so long as we were consistent for both Gaussians, but to keep things simple let's consider where each function falls to 1/*e* of its maximum value.

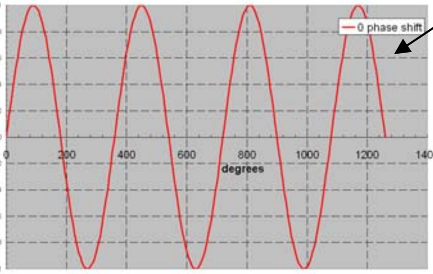
We can therefore say the 'half width' of *f(x)* is then $\Delta x_{1/2} = \sqrt{2/a}$ and for *F(k)* it is $\Delta k_{1/2} = \sqrt{2a}$. So the full widths are $\Delta x = 2\sqrt{2/a} = \sqrt{8/a}$ and $\Delta k = 2\sqrt{2a} = \sqrt{8a}$.

We find the following important result: **the product of the widths** $\boxed{\Delta x \Delta k = const = 8}$.

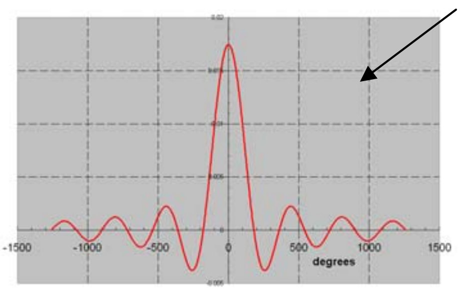
The product of the widths is therefore a *constant*, independent of *a*, its exact value determined by how the width is defined.

This is not just true for Gaussians. *It is true for any function and its Fourier transform*. (See tutorial questions!) The narrower the function, the wider the transform, and vice versa. The broader the function in real space (*x* space), the narrower the transform in *k* space. Or similarly, working with time and frequency, $\boxed{\Delta t \Delta \omega\ = constant}$.

One can understand this by thinking about 'wavepackets'. A pure sine wave *f(x)* = sin *kx* has uniform intensity throughout all space and comprises a single frequency, i.e. $\Delta x = \infty$, $\Delta k = 0$.



If we add together two sine waves of very similar *k, g(x)* = sin *kx* + sin (*k* + *δk)x,* the sines add together constructively at the origin but begin to cancel each other out (interfere destructively) further away. As one adds together more functions with a wider range of *k*'s (*Δk* increases), the waves add constructively over an increasingly narrow region (*Δx* decreases), and interfere destructively everywhere else.



The result above is related to the *uncertainty relationship* in quantum mechanics. Remember that momentum is related to *k* by $p = \hbar k$. Thus $\Delta p = \hbar \Delta k$.

An ideal 'free particle' can be represented by a wavefunction $\psi(x,t) = \psi_0 e^{i(kx - \omega t)}$, that is it has a definite value of *k*, a definite momentum. Correspondingly this wavefunction extends through all space – so we cannot say where the particle is! $\Delta k = \Delta p = 0$, $\Delta x = \infty$.

A particle which is localised in space (has finite *Δx*) must be represented by a 'wavepacket' with a spread of *k*, a spread of momenta. We have $\Delta x \Delta k \sim 1$, hence $\Delta x \Delta p \sim \hbar$. This is the uncertainty relationship between position and momentum.

In the same way, from $\Delta t \Delta \omega \sim 1$, and the relationship $E = \hbar \omega$ we have $\Delta E \Delta t \sim \hbar$. This is the uncertainty relation between energy and time.

Example 3: A pulse of radiation

Consider a pulse of light given by $f(t) = \begin{cases} A\cos\omega_0 t & -\tau < t < \tau \\ 0 & otherwise \end{cases}$.

The frequency spectrum is given by

$$F(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt = \frac{A}{\sqrt{2\pi}}\int_{-\tau}^{\tau}\cos\omega_0 t\, e^{-i\omega t}dt = \frac{A}{\sqrt{2\pi}}\int_{-\tau}^{\tau}\frac{1}{2}(e^{i\omega_0 t} + e^{-i\omega_0 t})e^{-i\omega t}dt$$

$$= \frac{A}{2\sqrt{2\pi}}\int_{-\tau}^{\tau} e^{i(\omega_0-\omega)t} + e^{-i(\omega_0+\omega)t}dt = \frac{A}{2\sqrt{2\pi}}\left[\frac{e^{i(\omega_0-\omega)t}}{i(\omega_0-\omega)} - \frac{e^{-i(\omega_0+\omega)t}}{i(\omega_0+\omega)}\right]_{-\tau}^{\tau}$$

$$F(\omega) = \frac{A}{2i\sqrt{2\pi}}\left[\frac{e^{i(\omega_0-\omega)t}}{(\omega_0-\omega)} - \frac{e^{-i(\omega_0+\omega)t}}{(\omega_0+\omega)}\right]_{-\tau}^{\tau} = \frac{A}{2i\sqrt{2\pi}}\left[\left\{\frac{e^{i(\omega_0-\omega)\tau}}{(\omega_0-\omega)} - \frac{e^{-i(\omega_0+\omega)\tau}}{(\omega_0+\omega)}\right\} - \left\{\frac{e^{-i(\omega_0-\omega)\tau}}{(\omega_0-\omega)} - \frac{e^{i(\omega_0+\omega)\tau}}{(\omega_0+\omega)}\right\}\right]$$

Grouping terms $F(\omega) = \frac{A}{2i\sqrt{2\pi}}\left[\left\{\frac{e^{i(\omega_0-\omega)\tau}}{(\omega_0-\omega)} - \frac{e^{-i(\omega_0-\omega)\tau}}{(\omega_0-\omega)} - \frac{e^{-i(\omega_0+\omega)\tau}}{(\omega_0+\omega)} + \frac{e^{i(\omega_0+\omega)\tau}}{(\omega_0+\omega)}\right\}\right]$

But remember that $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ so we can write

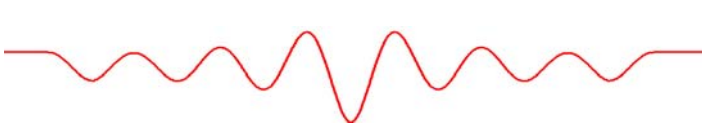
$$F(\omega) = \frac{A}{2i\sqrt{2\pi}}\left[\left\{\frac{2i\,\sin(\omega_0-\omega)\tau}{(\omega_0-\omega)} + \frac{2i\,\sin(\omega_0+\omega)\tau}{(\omega_0+\omega)}\right\}\right] = \frac{A}{\sqrt{2\pi}}\left[\left\{\frac{\sin(\omega_0-\omega)\tau}{(\omega_0-\omega)} + \frac{\sin(\omega_0+\omega)\tau}{(\omega_0+\omega)}\right\}\right]$$

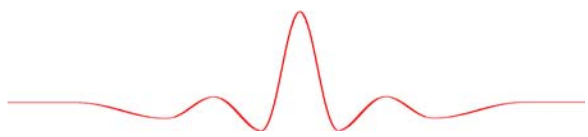
Now let's multiply both top and bottom by τ. $F(\omega) = \frac{A\tau}{\sqrt{2\pi}}\left[\left\{\frac{\sin(\omega_0-\omega)\tau}{(\omega_0-\omega)\tau} + \frac{\sin(\omega_0+\omega)\tau}{(\omega_0+\omega)\tau}\right\}\right]$

Often $\omega_0\tau >> 1$ so the second term is very small and we need only consider the first term:

$F(\omega) = \frac{A\tau}{\sqrt{2\pi}}\left[\left\{\frac{\sin(\omega_0-\omega)\tau}{(\omega_0-\omega)\tau}\right\}\right]$ The frequencies that are present are essentially those in the range $-\pi < (\omega_0-\omega)\tau < \pi$ i.e. $\omega = \omega_0 \pm \Delta\omega$ where $\Delta\omega = \pi/\tau$. So the full width of frequencies is $2\Delta\omega = 2\pi/\tau$.

Note that if the pulse is very long the frequency spread is very small – essentially the only frequency observed will be ω_0. This is as expected. But for a short pulse there will be significant broadening.





This result is relevant to **pulsed lasers**. Ti-sapphire lasers have been developed which give very short pulses of light – pulses lasting just a few femtoseconds. Such light pulses are used, for example, to probe relaxation phenomena in solids. However the frequency of the light itself is only a little greater than 10^{15}Hz so this means that we really have a short cosine wave pulse in time, and the frequency is therefore spread. While CW (continuous wave) lasers can emit light with an extremely narrow line-width, pulsed laser light must, by its very nature, have a broader line-width. And the shorter the pulse, the broader the line-width.

Example 4: The 'one-sided exponential' function

Show that the function $f(x)=\begin{cases}0 & x<0\\ e^{-\lambda x} & x>0\end{cases}$ has Fourier transform $F(k)=\dfrac{1}{\sqrt{2\pi}}\dfrac{1}{\lambda+ik}$.

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-\lambda x}e^{-ikx}dx=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}e^{-x(\lambda+ik)}dx=\frac{-1}{\sqrt{2\pi}}\frac{1}{\lambda+ik}\left[e^{-\infty}-e^{0}\right]=\frac{1}{\sqrt{2\pi}}\frac{1}{\lambda+ik}$$

4. Complexity, Symmetry and the Cosine Transform

The formula for *F(k)* involves e^{ikx}. So in general, if *f(x)* is real, *F(k)* will be complex.
In example 4, *F(k)* is complex. However in examples 1, 2 and 3, *F(k)* is *real*. Why?
For Fourier *series* we found special results for even and odd functions. It is similar for Fourier transforms.

We can write $F(k)=\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}f(x)e^{-ikx}dx=\dfrac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\cos kx\,dx-\dfrac{i}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\sin kx\,dx$

For an *even* function, the second integral will be zero (the integral of an odd function over a symmetric interval). Hence **if *f(x)* is real and even then its Fourier transform is real** (examples 1, 2, and 3).
Similarly for an *odd* function the first integral will be zero, so the Fourier transform is purely imaginary.
In the general case of a function with no definite symmetry, the Fourier transform is complex (example4).

For even functions, also $\int_{-X}^{X}f_e(x)dx=2\int_0^{X}f_e(x)dx$, so we can write: $F(k)=\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^{\infty}f_e(x)\cos kx\,dx$.

Such transforms *F(k)* are also even, so the inverse transform can be similarly simplified. So for an even real function we can write

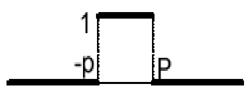
$$\boxed{f_e(x)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}F_e(k)\cos kx\,dx \quad \text{where} \quad F_e(k)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f_e(x)\cos kx\,dx}$$

F(k) is then called the **Fourier cosine transform** of *f(x)*.

The cosine integral is sometimes (but not always!) easier to evaluate.

Example 5: Repeat Example 1 using the Fourier cosine transform formula above.

$$F_e(k)=\sqrt{\frac{2}{\pi}}\int_0^{\infty}f_e(x)\cos kx\,dx=\sqrt{\frac{2}{\pi}}\int_0^{p}\cos kx\,dx=\sqrt{\frac{2}{\pi}}\left[\frac{1}{k}\sin kx\right]_0^{p}=\sqrt{\frac{2}{\pi}}\frac{1}{k}\sin kp$$



5. Use of Fourier Transforms

General Comments

- Given Fourier theory, we can understand why in optics and many other branches of physics one starts by saying, 'let us consider a plane wave'. If we can solve to find the behaviour of a plane wave we can use Fourier analysis to do *anything* by combining plane waves. For example, in an electronics problem one might want to know the response of a circuit to an electrical pulse. One can consider the response of the circuit to a single frequency and then use Fourier theory to write the pulse in terms of single frequencies.
- Computational Methods. A very efficient method has been devised for calculating the Fourier transform of a set of data points. It is known as '*fast Fourier Transforms*' or FFT. It is used very widely in many branches of computational physics. You may meet this in the future.

Physics/Astronomy Examples

5.1 Optics: Diffraction of Light

From school physics lessons, you may be familiar with the *diffraction* of light at small apertures. Many of you will study this in more detail next semester (PHY227). Consider a small slit illuminated uniformly by light of wavelength λ. At the slit, the light amplitude $f(x)$, and thus also the light intensity $|f(x)|^2$, will be similar to the 'top hat' function – example 1 above. Observing the diffraction pattern on a distant screen, the intensity at any point is given by the Fourier transform squared: $|F(k)|^2$ where $k = \frac{2\pi \sin\theta}{\lambda}$ and θ is the angular position on a distant screen. From example 1, the intensity at the screen therefore has a $sinc^2$ distribution – we observe a bright central fringe and regularly spaced side fringes of decreasing intensity.

Similarly in *all* cases of scattering, the intensity of the scattered light is given by the square of the FT of the object that does the scattering. Since light has a small wavelength, $\lambda \sim 3\times10^{-7}m$ we only get a reasonable range of θ values for a small object e.g. $d \le 10^{-6}m$. (In everyday life, you may be able to see a diffraction pattern by looking at a sodium streetlight through an umbrella or through a mist of raindrops.)

Fourier transforms go both ways, so also from looking at a diffraction pattern we can deduce the nature of the object causing the scattering. For example, crystal lattices can scatter X-rays and from the diffraction patterns the crystal structure can be deduced.

5.2 Nuclear Physics: scattering of electrons

Consider a beam of electrons scattered by protons. Full analysis requires relativistic quantum mechanics. But we expect the same features as in other scattering: the scattering will be largest for $kd < \pi$ and very small for $kd >> \pi$. The dependence of the scattering on k is known as the form factor. Some decades ago, protons were thought to be elementary particles, so we would expect to find d to be of the order of the size of small nuclei. However the data does not fit the predictions!! Remember that a broader $F(k)$ indicates a narrower $f(x)$. This data was the first evidence for quarks and gluons!

5.3 Telecommunications: bandwidth limitations

As we have seen in example 3, the shorter the pulse, the broader is the frequency distribution in the Fourier series required to describe it. The telecommunications industry constantly try to improve data transfer rate along cables. Typically the data takes the form of a digital signal and improved speed is achieved by shortening the lengths of the 1s and 0s. This extends the frequency distribution of the Fourier series that describes it. If the bandwidth limit of a telephone cable is 10MHz then only frequencies below 10MHz can pass, effectively clipping the high frequency end of the data signal and deforming the shape of the logic pulse square wave a little. At some point this will limit data transfer.

6. Delta Functions

The Dirac delta function $\delta(x)$ is very useful in many areas of physics. It is not an ordinary function, in fact properly speaking it can only live *inside* an integral. Here we define it, explore its properties, then use it to prove the Fourier integral theorem.

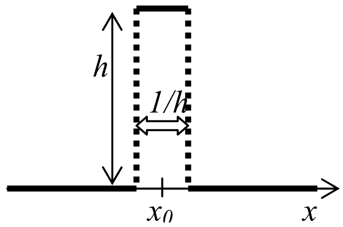
Essentially the delta function is an infinitely narrow 'spike' that has unit area.
$\delta(x)$ is a spike centred at $x = 0$, $\delta(x - x_0)$ is a spike centred at $x = x_0$.
The product of the delta function $\delta(x - x_0)$ with any function $f(x)$ is zero except where $x \sim x_0$.

Formally, for any function $f(x)$ $\boxed{\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = f(x_0)}$ and $\boxed{\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1}$.

In fact these integrals can also be written $\int_a^b f(x)\delta(x-x_0)\,dx = f(x_0)$ and $\int_a^b \delta(x-x_0)\,dx = 1$
where $a < x_0 < b$, since $\delta(x - x_0)$ is also defined to be zero everywhere except at $x = x_0$.

There are several ways in which we picture $\delta(x - x_0)$.

The simplest is as the limit as $h \to \infty$ of a rectangle of height h, width $1/h$, and thus area $h/h = 1$.



Examples

Given that $\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = f(x_0)$,

(a) find $\int_{-\infty}^{\infty} \sin x\, \delta(x-x_0)\,dx$, (b) find $\int_{-\infty}^{\infty} x^2\delta(x-a)\,dx$, (c) find the FT of $f(x) = \delta(x - a)$.

(a) (b) (c)

7. Proof of the Fourier Transform Theorem

The formulae for Fourier transforms can be derived from the formulae for Fourier series by considering a periodic function then letting the period tend to infinity. You can find this type of derivation in *Kreyszig* (section 11.7), *Jordan & Smith* (section 27.1 – note that the authors work in terms of *f* not ω and show the proof for sine series / integrals) and *Boas* (section 15.4 – uses the complex form but note their different convention regarding factors of 2π).

Consider the integral $I(x) = \frac{1}{2\pi}\int_{-L}^{L} e^{ikx}\,dk$.

We have $I(x) = \frac{1}{2\pi}\int_{-L}^{L} e^{ikx}\,dk = \frac{1}{2\pi ix}\left[e^{ikx}\right]_{-L}^{L} = \frac{1}{2\pi ix}(e^{iLx} - e^{-iLx}) = \frac{\sin Lx}{\pi x} = \frac{L}{\pi}\frac{\sin Lx}{Lx}$ (cf. example 1).

This sinc function tends to L/π as $x \to 0$ and away from there becomes small.

We stated earlier that $\int_{-\infty}^{\infty} dx \frac{\sin \pi x}{\pi x} = 1$. Similarly $\int_{-\infty}^{\infty} \frac{\sin Lx}{\pi x}\,dx = 1$.

So we see that taking the limit as $L \to \infty$ of $I(x)$ we have the properties we want in a delta function.

So we can define a delta function as $\delta(x) = \lim_{L\to\infty} \frac{1}{2\pi}\int_{-L}^{L} e^{ikx}\,dk$.

Hence $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\,dk$ or $\delta(x - x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-x_0)}\,dk$.

We will use this definition to prove the Fourier integral theorem.

Earlier we stated $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk$ where $F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$.

Given a function *f(x)*, the second formula defines *F(k)*. What needs to be proved is that substituting this *F(k)* into the first formula does yield the original function *f(x)*.

Put the second integral into the first:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(k)e^{ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x')e^{-ikx'}\,dx'\right]e^{ikx}\,dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x')e^{-ikx'}e^{ikx}\,dx'\,dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x')\,dx'\int_{-\infty}^{\infty} e^{-ik(x-x')}\,dk$$

$$= \int_{-\infty}^{\infty} f(x')\,\delta(x - x')\,dx' = f(x)$$

Hence, using our definition of a delta function, we have proved the theorem.