Lecture 4: 2nd Order ODE's cont.

INTRO: Hopefully these equations from PHY102 Waves & Quanta are familiar to you….

Free Oscillation with damping:

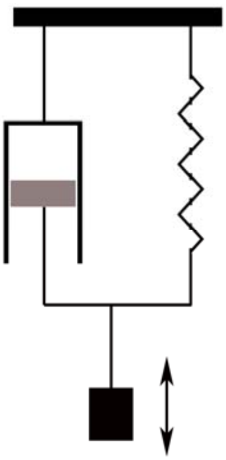
$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0$$

Forced Oscillation with damping:

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=H_0\cos\omega_D t$$

In this lecture we consider one more common homogeneous equation then two inhomogeneous equations.

Example 3. The Damped Harmonic Oscillator $\dfrac{d^2x}{dt^2}+2\gamma\dfrac{dx}{dt}+\omega_0^2x=0$



Looking for solutions of the form e^{mt} we obtain the characteristic equation $m^2+2\gamma m+\omega_0^2=0$.

This quadratic has two solutions: $m=-\gamma\pm\sqrt{\gamma^2-\omega_0^2}$ ***Be careful!*** There are **three** different cases.

(i) $\gamma^2>\omega_0^2$ (over-damping)

We have two real values for m: $m_1=-\gamma+\sqrt{\gamma^2-\omega_0^2}$ and $m_2=-\gamma-\sqrt{\gamma^2-\omega_0^2}$.

And the general solution is $x(t)=Ae^{m_1t}+Be^{m_2t}$.

Both m_1 and m_2 are negative so $x(t)$ is the sum of two exponential decay terms and so tends pretty quickly, to zero. The effect of the spring has been made of secondary importance to the huge damping, e.g. fire doors.

(ii) $\gamma^2=\omega_0^2$ ('critical damping')

The characteristic equation has a double root $m=-\gamma$, so the general solution is $x(t)=e^{-\gamma t}[A+Bt]$ as shown earlier. Here the damping has been reduced a little so the spring can act to change the displacement quicker. However the damping is still high enough that the displacement does not pass through the equilibrium position, e.g. car suspension – push down on the wheel arch and hope not to see SHM!

(iii) $\gamma^2<\omega_0^2$ (under-damping)

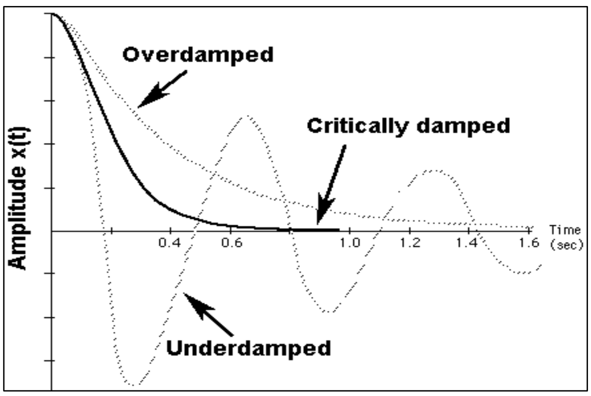
The roots are complex. Define $\Omega^2=\omega_0^2-\gamma^2$ so $\sqrt{\omega_0^2-\gamma^2}=\pm\Omega$ and $\sqrt{\gamma^2-\omega_0^2}=\pm i\Omega$.

Then the two allowed values of m can be written $m_1=-\gamma+i\Omega$ and $m_2=-\gamma-i\Omega$.

The general solution can be written $x(t)=e^{-\gamma t}[Ae^{i\Omega t}+Be^{-i\Omega t}]$

or $x(t)=e^{-\gamma t}[C\cos\Omega t+D\sin\Omega t]$ or $x(t)=Fe^{-\gamma t}\cos(\Omega t+\phi)$. See Phil's Problems Lect3Prob6.

The solution is the product of a sinusoidal term and an exponential decay term – so represents sinusoidal oscillations of decreasing amplitude. E.g. bed springs.



The amplitude will fall to $1/e$ of its original value after a time $\tau=\dfrac{1}{\gamma}$.

In many physically interesting cases $\gamma^2<<\omega_0^2$. In this case $\Omega\sim\omega_0$, so $x(t)\approx Fe^{-\gamma t}\cos(\omega_0 t+\phi)$.

In that time τ the oscillator will have made n oscillations where $n=f\tau$ and $f=\dfrac{\omega_0}{2\pi}$ hence $n=\dfrac{\omega_0}{2\pi\gamma}$. The ratio $\omega_0/2\gamma$ is called Q, the *quality factor*. Q is widely used in all areas of physics, a higher Q indicating a lower rate of energy dissipation relative to the oscillation frequency, so oscillations die more slowly. (see PHY102 topic 1 and PHY221).

Inhomogeneous Equations

We now look at *inhomogeneous* or *forced* or *driven* second order linear ODEs with constant coefficients.

These are equations of the form $\boxed{a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f(t)}$.

The two common driven equations which we will discuss are:

Example 4.	$\frac{d^2}{dt^2}x(t)+\omega_0^2x(t)=F\cos\omega t$	Driven oscillator no damping
Example 5.	$\frac{d^2}{dt^2}x(t)+2\gamma\frac{dx(t)}{dt}+\omega_0^2x(t)=F\cos\omega t$	Driven oscillator with damping

Equation 5 has applications in countless different areas of science! These include mechanical oscillators, LCR circuits, optics and lasers, NMR, nuclear physics, Mössbauer effect, pulsars, etc. etc. Equation 4 is usually unphysical, but it's much easier to solve, so we will look at this first!

Revision of Theory Solution involves four steps:

1) Find the general solution of the related homogeneous equation $a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=0$ (by the methods discussed earlier). Call this complementary solution $x_c(t)$.
2) Find any solution of the full equation. This solution, $x_p(t)$, is often called a *particular solution* or *particular integral.* It is found using an appropriate trial solution.

e.g.	If	$f(t) = t^2$	try	$x_p(t) = at^2 + bt + c$
	If	$f(t) = 5e^{3t}$	try	$x_p(t) = ae^{3t}$
	If	$f(t) = 5e^{i\omega t}$	try	$x_p(t) = ae^{i\omega t}$
	If	$f(t) = \sin 2t$	try	$x_p(t) = a\cos 2t + b\sin 2t$ (or complex version – see below!)
	If	$f(t) = \cos wt$	try	$x_p(t) = \mathrm{Re}[ae^{i\omega t}]$ see later for explanation
	If	$f(t) = \sin wt$	try	$x_p(t) = \mathrm{Im}[ae^{i\omega t}]$

   If your trial solution has the correct form, substituting it into the differential equation will yield the values of the constants *a, b, c,* etc.
3) The complete general solution is the sum of the two parts above, $x = x_c + x_p$.
4) The complete general solution contains two constants (in x_c). If two boundary conditions are known, these should be applied to find the values of the constants.

Example 4. The Undamped, Driven Oscillator $\frac{d^2}{dt^2}x(t)+\omega_0^2x(t)=F\cos\omega t$

Step 1 The corresponding homogeneous equation is simply the LHO equation. From the last lecture, therefore, we can take, say, $x_c(t)=A\cos\omega_0 t+B\sin\omega_0 t$.

Step 2 We need to find the 'particular integral' using a trial solution. We should try $x_p(t) = a\cos\omega t + b\sin\omega t$. Substitute this trial solution into the original equation:

We find $(\omega_0^2 - \omega^2)a\cos\omega t + (\omega_0^2 - \omega^2)b\sin\omega t = F\cos\omega t$.

Comparing terms we can say that b = 0 and…. $(\omega_0^2 - \omega^2)a = F$

Hence the trial solution is a solution provided $a=\frac{F}{\omega_0^2-\omega^2}$, i.e. $x_p(t)=\frac{F}{\omega_0^2-\omega^2}\cos\omega t$.

Step 3 So the complete general solution is $x(t)=A\cos\omega_0 t+B\sin\omega_0 t+\frac{F}{\omega_0^2-\omega^2}\cos\omega t$

Step 4 Suppose a particle subject to the equation above is known to be at rest at $x = L$ at $t = 0$.

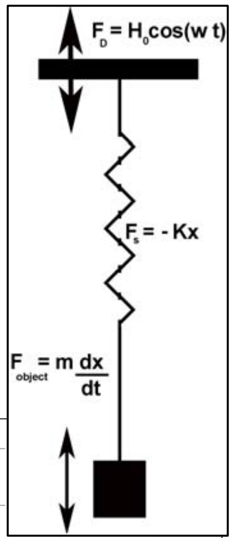
This means we have the boundary conditions $x(0) = L$ and $\frac{dx}{dt}|_{t=0} = 0$.

Substitute t = 0 in the general solution given above: $x(0) = A + 0 + \frac{F}{\omega_0^2 - \omega^2} = L$

Differentiating the general solution, *then* substituting $t = 0$ gives $\omega_0 B = 0$

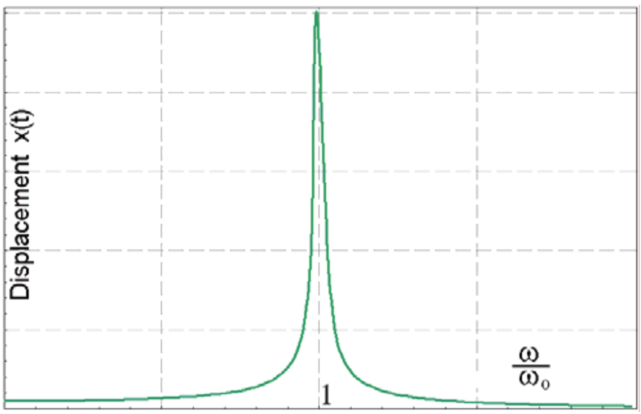
Hence B = 0 and $A = L - \frac{F}{\omega_0^2 - \omega^2}$ so the solution is: $x(t) = (L - \frac{F}{\omega_0^2 - \omega^2})\cos\omega_0 t + \frac{F}{\omega_0^2 - \omega^2}\cos\omega t$

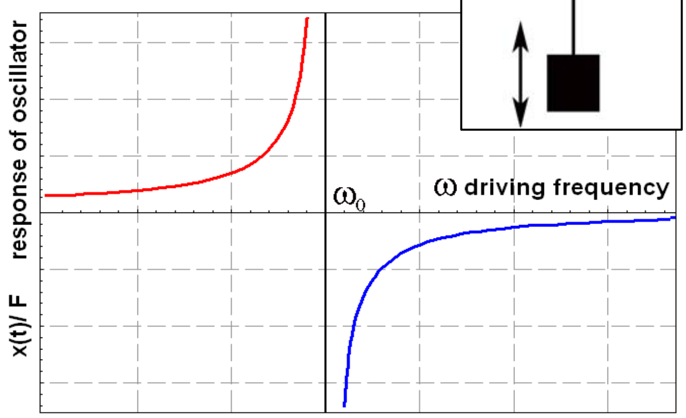
This can be written as $x(t) = L\cos\omega_0 t + \frac{F}{(\omega_0^2 - \omega^2)}(\cos\omega t - \cos\omega_0 t)$.



A few comments

1. Note that the solution is clearly not valid for $\omega = \omega_0$!
2. The ratio $\frac{x(t)}{F}$ is sometimes called the response of the oscillator. It is a function of ω. It is positive for $\omega < \omega_0$, negative for $\omega > \omega_0$. This means that at low frequency the oscillator follows the driving force but at high frequencies it is always going in the 'wrong' direction.





Solution using Complex Numbers

The particular integral of the equation above was easy to find because a trial function of the form $x_p(t) = a\cos\omega t + b\sin\omega t$ worked. In our next equation (a driven oscillator with damping) this trial function would also work ... but the algebra gets very messy. It is easier to use complex numbers. To learn the complex method we will use it to solve equation 4 again for the particular integral.

Compare the original equation $\frac{d^2}{dt^2}x(t) + \omega_0^2 x(t) = F\cos\omega t$ (A)

With the equation $\frac{d^2}{dt^2}X(t) + \omega_0^2 X(t) = Fe^{i\omega t}$ (B)

We know $F\cos\omega t = Re(Fe^{i\omega t})$, so *if equation (B) has (complex) solutions X(t) then the solutions of equation (A) will be the real part of these: x(t) = Re(X(t)).* If the function on the RHS of (A) was $\sin\omega t$ then we could use the same approach but at the end take the imaginary part.

i.e. first we solve $\frac{d^2}{dt^2}X(t) + \omega_0^2 X(t) = Fe^{i\omega t}$.

This is easy: we take a trial solution of the form $X = Ae^{i\omega t}$.

Substituting this in gives: $(\frac{d^2}{dt^2} + \omega_0^2)Ae^{i\omega t} = (-\omega^2 + \omega_0^2)A(\omega)e^{i\omega t} = Fe^{i\omega t}$

Hence $A(\omega) = \frac{F}{(\omega_0^2 - \omega^2)}$ so $X(t) = \frac{F}{(\omega_0^2 - \omega^2)}e^{i\omega t}$

<u>Finally last thing we do</u> is take the real part: $x(t) = Re(X(t)) = \dfrac{F\cos\omega t}{(\omega_0^2 - \omega^2)}$ (as before).

<u>Example 5.</u> The Damped, Driven Oscillator $\dfrac{d^2}{dt^2}x(t) + 2\gamma\dfrac{dx(t)}{dt} + \omega_0^2 x(t) = F\cos\omega t$

Step 1 The complementary function will be the solution of the damped harmonic oscillator, found at the beginning of this lecture. As discussed there, the appropriate form depends on the magnitude of γ compared to ω_0. However note that in every case, the solution tends to zero as t $\to\infty$. It is often called the "transient" solution.

Step 2 The <u>particular integral</u>, by contrast, does not die away and is called the "steady state solution". We will find it using the complex method described above.

Consider the equation $\dfrac{d^2}{dt^2}X(t) + 2\gamma\dfrac{dX(t)}{dt} + \omega_0^2 X(t) = F\cos\omega t = Fe^{i\omega t}$.

Look for solution of form $X = A(\omega)e^{i\omega t}$: $(\dfrac{d^2}{dt^2} + 2\gamma\dfrac{d}{dt} + \omega_0^2)Ae^{i\omega t} = (-\omega^2 + 2i\omega\gamma + \omega_0^2)Ae^{i\omega t} = Fe^{i\omega t}$

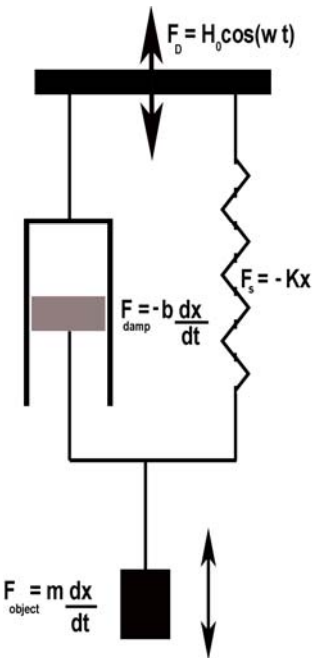
So $A(\omega) = \dfrac{F}{(-\omega^2 + 2i\omega\gamma + \omega_0^2)} = \dfrac{F}{Z(\omega)}$. Remember to divide by a complex, we write it in form $re^{i\phi}$.

Let $(-\omega^2 + 2i\omega\gamma + \omega_0^2) = Z(\omega) = |Z(\omega)|e^{i\phi}$ where $|Z(\omega)| = \sqrt{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$ and $\tan\phi = \dfrac{2\gamma\omega}{\omega_0^2 - \omega^2}$.

Then $A(\omega) = \dfrac{F}{Z(\omega)} = \dfrac{Fe^{-i\phi}}{|Z(\omega)|}$ so $X = \dfrac{Fe^{-i\phi}}{|Z(\omega)|}e^{i\omega t} = \dfrac{F}{|Z(\omega)|}e^{i(\omega t - \phi)}$, and now the **<u>last thing we do</u>** is to take the real part of the answer; hence $x(t) = \mathrm{Re}[X] = \mathrm{Re}[\dfrac{F}{|Z(\omega)|}e^{i(\omega t-\phi)}]$ so $x(t) = \dfrac{F}{|Z(\omega)|}\cos(\omega t - \phi)$.

(**Steps 3 & 4** can then be followed if required.)

In cases where the damping is small, the amplitude has a strong peak at $\omega \approx \omega_0$ and the quality factor Q is again an important indicator.



Closing remarks

We have focussed on the *mathematics* of solving generic harmonic oscillator equations. By replacing ω, γ, etc. with appropriate constants, you should now be able to solve equations for all mechanical oscillators, oscillations in electrical LCR circuits, and numerous other oscillators! PHY221 and other courses will explore more of the physical significance of the solutions found here.

References

The material of lectures 3&4 is covered very thoroughly, with many real physical examples, by *French* in the course pack p.5-52:

Undamped, undriven LHO	7-9
Damped*, undriven LHO & Q-factor	10-16
Undamped, driven LHO: steady state	20-24
… again using complex exponentials	24-25
Damped*, driven LHO: steady state	25-28
Further discussion of Q, transients, resonance, etc.	31-42
Electrical, optical & nuclear examples	42-52

[*Note that French uses a damping constant γ while we have used 2γ]