Ordinary Differential Equations (ODE's) I

Most of physics involves the solution of differential equations! The solution of *ordinary* differential equations (ODEs) was covered in PHY112. 'Ordinary' means that all functions are of only one variable. We will revise the theory and explore some examples, especially harmonic oscillators. Later lectures will address the solution of *partial* differential equations featuring multiple variables.

First Order ODEs (i.e. 1 variable and no higher than $\frac{dx}{dt}$ terms)

Revision of Theory

You should be aware of two possible methods for solving 1st order ODEs. Which method you use depends on the equation you are trying to solve.

1. Some equations can be solved by the method of **separation of the variables**: rearrange the equation so that each side involves only one variable, then integrate both sides.
2. The method of **trial solution** may be used.

The **general solution** of a 1st order equation will contain one *arbitrary constant*; the value of the constant is determined by the *boundary conditions*, yielding a **particular solution**.

Example: Radioactive Decay

Consider a sample of radioactive material. Let N be the number of undecayed atoms at time t.

At any time, the rate at which atoms decay is proportional to N. I.e. $\frac{dN(t)}{dt} = -\lambda N(t)$ where λ is the decay constant. Given that $N = N_0$ at $t = 0$, find an expression for N at later times.

Method 1

a) $\frac{dN}{N} = -\lambda dt$ can be rearranged and both sides integrated: $\int \frac{dN}{N} = -\lambda \int dt$.

Performing these (indefinite) integrals we obtain $\ln N = -\lambda t + c$ (*remember the constant!)*

Hence $N = e^{-\lambda t + c} = e^{-\lambda t} e^{c} = Ae^{-\lambda t}$ where $A = e^{c}$.

Using the boundary condition that at $t = 0$, $N = N_0$, we find $A = N_0$. Hence $N(t) = N_0 e^{-\lambda t}$.

b) Alternatively the boundary condition information can be entered as the limits of definite integrals:

$$\int_{N_0}^{N} \frac{dN}{N} = -\lambda \int_0^t dt \quad \text{giving} \quad \ln N - \ln N_0 = \ln \frac{N}{N_0} = -\lambda t \quad \text{, hence} \quad N(t) = N_0 e^{-\lambda t}.$$

Method 2

We may guess that the equation has a solution of the form $N(t) = Ae^{mt}$.

Substituting this trial solution into the equation gives $\frac{dN(t)}{dt} = mN(t) = -\lambda N(t)$.

So it is a solution if $m = -\lambda$. i.e. the general solution is $N(t) = Ae^{-\lambda t}$.

Applying the boundary condition we find the solution as before.

Example 1: The growth of an ant colony is proportional to the number of ants. If at t = 0 days there are only 2 ants, but after 20 days there are 15 ants, what is the differential equation and what is its solution?

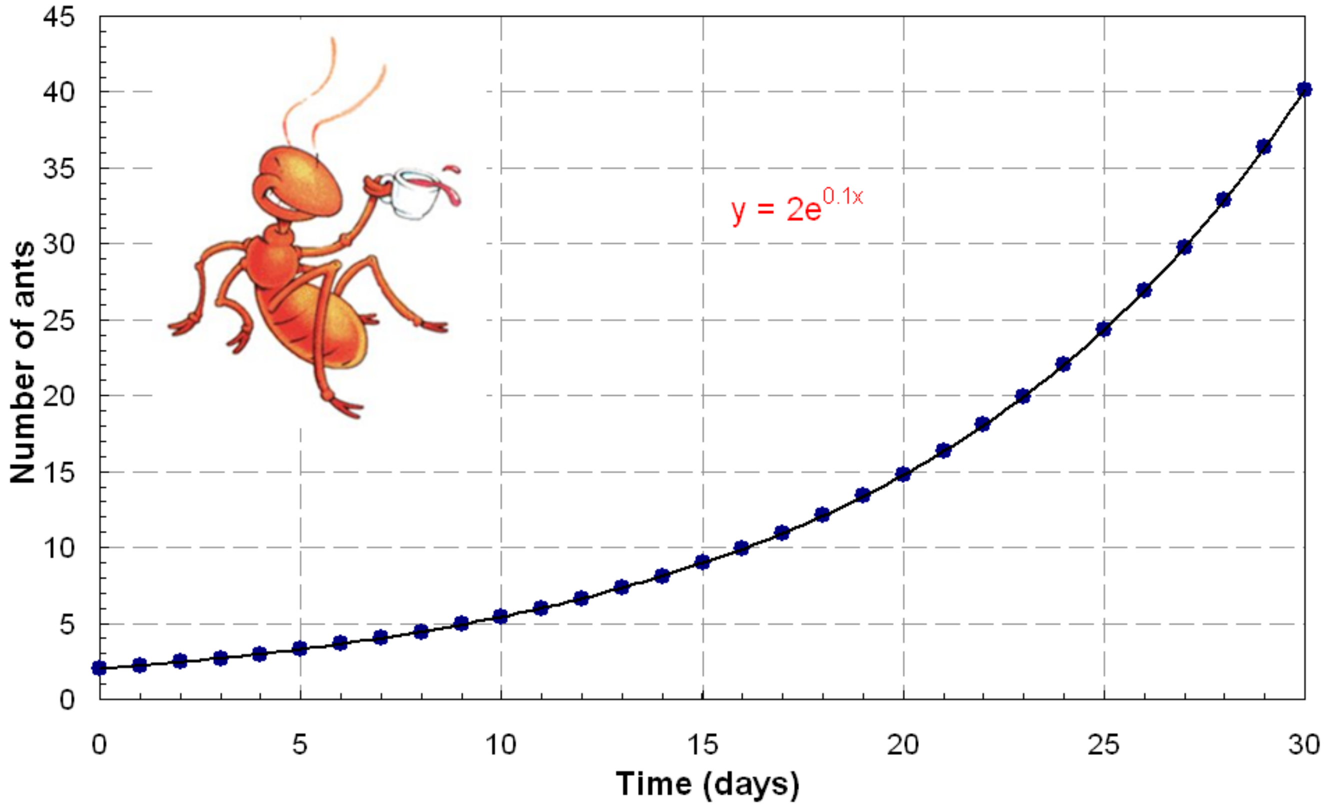
The differential equation is $\frac{dP}{dt} = KP$ where the population at time t is P(t).

$\frac{dP}{P} = Kdt$ and then integrate to get $\ln P = Kt + c$ otherwise written as $P = \exp(Kt + c) = A\exp(Kt)$

We are told that at t = 0, P = 2 so …. 2 = A

We are also told that at 20 days P =15 so ….. 15 = 2 exp (20K), so K = 0.1

The solution is therefore $P = 2\exp(0.1t)$



Second Order ODEs

We will restrict our study of 2nd order ODEs to that of *linear equations with constant coefficients*

These are equations of the form $a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=f(t)$.

We look first at equations with $f(t)=0$, called *homogeneous* or *unforced*.
Next lecture we look at equations with $f(t)\neq 0$, called *inhomogeneous* or *forced* or *driven*.
[*Note:* In this course we concentrate on the mathematics; the physics is further explored in PHY221.]

Homogeneous Equations - Revision of Theory

We have the equation $a\frac{d^2x}{dt^2}+b\frac{dx}{dt}+cx=0$.

Looking for trial solutions of the form $x=e^{mt}$ leads to the or *auxiliary equation*

$$am^2+bm+c=0.$$

Find the roots of this equation.

- For real, distinct roots, m_1 and m_2, the general solution is $x=Ae^{m_1t}+Be^{m_2t}$
- For real, repeated roots, m, the general solution is $x=(At+B)e^{mt}$
- For complex roots $m=\alpha\pm i\beta$, the general solution may be written $x=Ae^{(\alpha+i\beta)t}+Be^{(\alpha-i\beta)t}$ $=e^{\alpha t}(Ae^{i\beta t}+Be^{-i\beta t})$ or equivalent form such as $x=e^{\alpha t}(C\sin\beta t+D\cos\beta t)=Ee^{\alpha t}[\cos(\beta t+\phi)]$.

NB. Proofs of these equivalent relationships can be found in Phil's Problems.

Note that the general solution contains *two* arbitrary constants. *Two* boundary conditions must therefore be applied to find a particular solution.

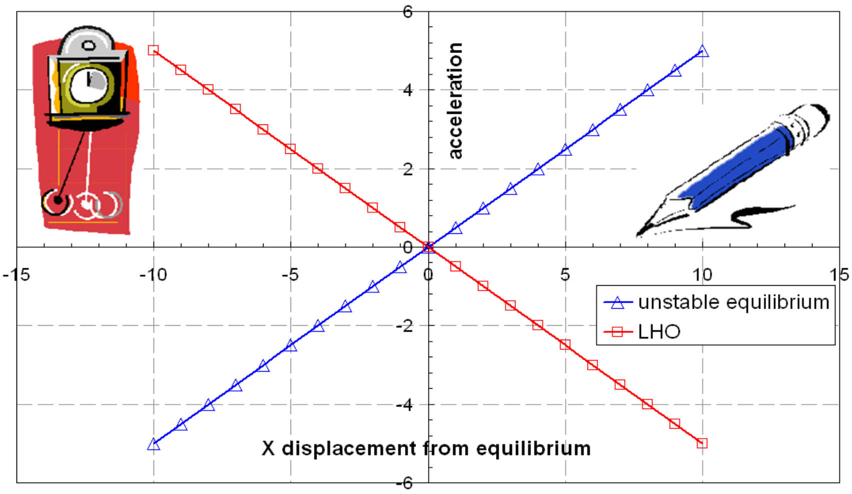
Homogeneous Equations – Simplest examples with no friction

Two forms which occur very commonly in physics are:

1. $\frac{d^2}{dt^2}x(t)=-\omega_0{}^2x(t)$ Linear harmonic oscillator (LHO) (Equivalently $\frac{d^2}{dt^2}x(t)+\omega_0^2x(t)=0$.)

This equation occurs almost everywhere! E.g. all problems concerning waves (strings, light, etc.); small oscillations e.g. lattice vibrations in solids; LC electric circuits.

2. $\frac{d^2}{dt^2}x(t)=\alpha^2x(t)$ Unstable equilibrium. Less common occurrences as most systems in unstable equilibrium collapse…..e.g. pencil balancing on its point.



<u>Example 1</u>. The Linear Harmonic Oscillator

Find the solution of $\frac{d^2}{dt^2}x(t) + \omega_0^2 x(t) = 0$?

Substituting $x = e^{mt}$ yields the auxilliary equation $m^2 + \omega_0^2 = 0$. Hence $m^2 = -\omega_0^2$, $m = \pm i\omega_0$.
We have the case of complex roots. In this case, the roots are pure imaginary. The general solution can be written in various different (but equivalent) forms:

(a) $x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}$, (b) $x(t) = C\sin\omega_0 t + D\cos\omega_0 t$, (c) $x = E[\cos(\omega_0 t + \phi)]$

We use whichever form of the solution is the most convenient, e.g. (b) for a standing wave and (a) for a travelling wave and (c) for a real oscillating wave where the total amplitude is important.

Applying Boundary Conditions

If the particle starts at the origin with velocity V, i.e. $x(0) = 0$ and $\frac{dx(t)}{dt}|_{t=0} = V$. Apply boundaries?

In this case a trig form is more familiar. We could use the form $x(t) = C\sin\omega_0 t + D\cos\omega_0 t$ (*)

Then the first condition gives $x(0) = 0 = \mathrm{D} + 0$ hence $D = 0$.

To use the 2nd condition we differentiate (*) *then* substitute t = 0, giving $V = \omega_0 C\cos(0) = \omega_0 \mathrm{C}$.

Hence in this case $D = 0$ and $C = \frac{V}{\omega_0}$ so the solution is $x(t) = \frac{V}{\omega_0}\sin\omega_0 t$.

<u>Example 2.</u> Unstable Equilibrium

Find the solution of $\frac{d^2}{dt^2}x(t) = \alpha^2 x(t)$?

The auxillary equation is $m^2 = \alpha^2$ so $m = \pm\alpha$. Hence the general solution is $x(t) = Ae^{\alpha t} + Be^{-\alpha t}$.

Applying Boundary Conditions

Suppose $x(0) = L$ and $\frac{dx(t)}{dt}|_{t=0} = 0$. Apply the boundary conditions?

The general solution is $x(t) = Ae^{\alpha t} + Be^{-\alpha t}$. So first condition gives $x(0) = \mathrm{L} = \mathrm{A} + \mathrm{B}$.

The second condition gives $\frac{dx(t)}{dt}|_{t=0} = 0 = \alpha Ae^{\alpha t} - \alpha Be^{-\alpha t} = \alpha A - \alpha B$. So A – B = 0. So A = B = $\frac{L}{2}$.

The solution is $x(t) = \frac{L}{2}(e^{\alpha t} + e^{-\alpha t}) = L\cosh\alpha t$.

Compare the solutions of equations (1) and (2). They have very different physical characteristics!

- Solutions of (1) oscillate for ever.
- Solutions of (2) grow to infinity as t increases.