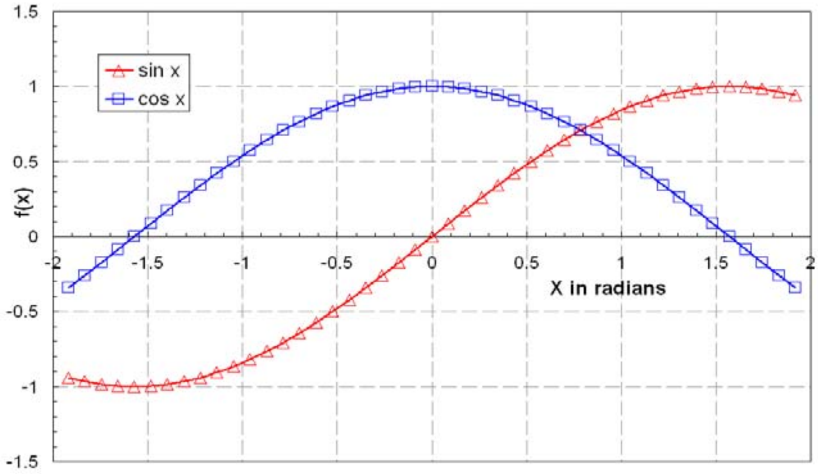
EXPONENTIALS, FUNCTIONS AND COMPLEX NUMBERS

A key observation looking at sin(x) and cos(x) below is that cos(x) is symmetrical in the Y-axis whereas sin(x) is not. This can be written mathematically as:-

cos(x) = cos(-x) **and** **sin(x) = - sin(-x)** **These relationships are crucial in this course**

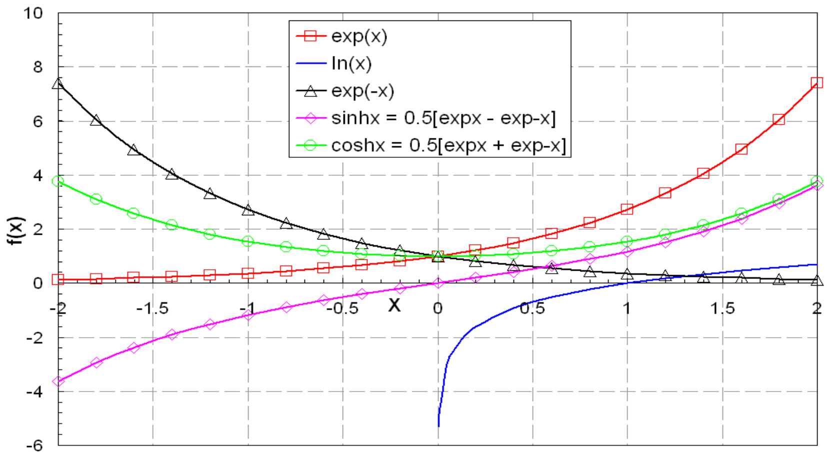


We can combine exponentials into the *hyberbolic* functions:

$$\cosh\alpha x = \frac{1}{2}\left[e^{\alpha x} + e^{-\alpha x}\right] \quad \text{is an } \textit{even function}\text{:} \quad \cosh\alpha x = \cosh(-\alpha x)$$

and

$$\sinh\alpha x = \frac{1}{2}\left[e^{\alpha x} - e^{-\alpha x}\right] \quad \text{is an } \textit{odd function}\text{:} \quad \sinh\alpha x = -\sinh(-\alpha x)\,.$$



Using the definition of the derivative it is easy to show that:

$$\frac{d\cosh\alpha x}{dx} = \frac{1}{2}\left[\frac{de^{\alpha x}}{dx} + \frac{de^{-\alpha x}}{dx}\right] = \frac{\alpha}{2}\left[e^{\alpha x} - e^{-\alpha x}\right] = \alpha\sinh\alpha x \quad \text{and} \quad \frac{d\sinh\alpha x}{dx} = \alpha\cosh\alpha x\,.$$

Exponentials are powers and so they satisfy: $\boldsymbol{e^{a+b} = e^a e^b}$ **and** $\boldsymbol{e^{-a} = 1 / e^a}$.

Natural logarithms are defined by $y = e^x \qquad x = \ln y$.

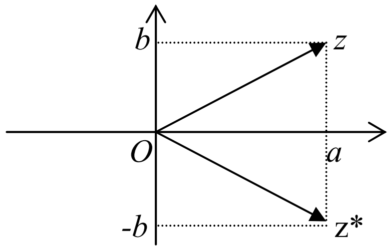
We also have that $\mathbf{\ln y_1 + \ln y_2 = \ln (y_1 y_2)}$ and $\mathbf{\ln y_1 - \ln y_2 = \ln (y_1/y_2)}$

We can relate natural logs to those to base 10:
Define $w = \log_{10} y$. This expression *means* that $y = 10^w$.

Take natural logs of both sides: $\ln y = \ln\left(10^w\right) = w\ln(10) \qquad w = \dfrac{\ln y}{\ln(10)} \quad \text{or} \quad y = e^{w\ln(10)}$.

Complex Numbers

Let $\boldsymbol{z = a + ib}$ where $i^2 = -1$ (Note: Physicists usually use i, engineers often use j.)
To represent this number on an **Argand diagram**, plot the point with Cartesian coordinates (a, b). i.e. real numbers run along the x axis and imaginary numbers along the y axis.

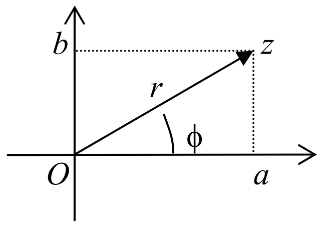


The **complex conjugate** is $\boldsymbol{z^* = a - ib}$
(Note: physicists *always* denote complex conjugates by z* *not* $\bar{z}$.)

By Pythagoras, the length OZ is $(a^2 + b^2)^{1/2}$.
Note that this length is also equal to $(z\,z^*)^{1/2}$, since $z\,z^* = (a+ib)(a-ib) = a^2 + (ib)(-ib) = a^2 + b^2$.
We also write $a^2 + b^2 = \boldsymbol{(z\,z^*) = |z|^2}$ where $|z| = \sqrt{zz^*}$ and is called the modulus of z.

<u>Example 1</u>: Find the modulus of |(2 + 3i)| ?
$|(2 + 3i)|^2 = (2 + 3i)(2 - 3i) = 4 - 9i^2 = 4 + 9 = 13.$ So $|(2 + 3i)| = \sqrt{13}$.

Polar form
We can also write $\boldsymbol{z = r\,e^{i\phi} = r\,(\cos\phi + i\sin\phi)}$, $0 < \phi < 2\pi$
where r = is again called the modulus, ϕ is called the *argument* or *phase*.
For a proof of this relationship see Lecture 2, problem 2 in Phil's Problems.



Then $z^* = r\,e^{-i\phi}$
So $zz^* = r^2 e^{i\phi} e^{-i\phi} = r^2$ since $e^{i\phi} e^{-i\phi} = e^{i(\phi-\phi)} = e^0 = 1$

Clearly $a = r\cos\phi$, $b = r\sin\phi$, and $r = \sqrt{a^2 + b^2} = |z|$.

Note that $|z|$ and $zz^* = |z|^2$, are always *real*, whereas $z^2 = a^2 + 2iab - b^2 = r^2 e^{2i\phi} \neq |z|^2$ is usually *complex*.
In physics we *always* need to get real answers, hence in quantum mechanics etc. one takes $|\psi|^2$ not ψ^2.
(In optics and E&M you may sometimes take the real part to get your answer.)

Changing between the forms $\boldsymbol{z = a + ib}$ and $\boldsymbol{z = re^{i\phi}}$
You are strongly advised to first *plot the number on an Argand diagram*. Without this it is easy to make mistakes about minus signs and angles, etc.!

- Given $z = a + ib$, to find the form $z = re^{i\phi}$: Find r using $r = \sqrt{a^2 + b^2}$.

  Find ϕ using $\tan\phi = \frac{b}{a}$ and specifying the quadrant, or read the angle off the Argand diagram.
- Given $z = re^{i\phi}$, to find the form $z = a + ib$ is easier: $a = r\cos\phi$ and $b = r\sin\phi$.

Why do we need both forms?
It is easier to *add* and subtract complex numbers in the form $z = a + ib$ but easier to *multiply, divide,* take *powers* and *roots* when they are in the form $z=re^{i\phi}$. In physics we almost always use the form $z = re^{i\phi}$.

Addition and Subtraction
If $z = a + ib$ and $w = c + id$ then $z + w = (a + c) + i(b + d)$ and $z - w = (a - c) + i(b - d)$.

Multiplication and Division
For this we *always* use the form $z = r\,e^{i\phi}$.
Let $z_1 = r_1 e^{i\phi_1}$ and $z_2 = r_2 e^{i\phi_2}$ then $z_1 z_2 = r_1 e^{i\phi_1} r_2 e^{i\phi_2} = r_1 r_2 e^{i(\phi_1+\phi_2)}$
i.e. *multiply* the moduli and *add* the arguments (phases).
Similarly for division: $(z_1 / z_2) = (r_1 / r_2)\, e^{i(\phi_1-\phi_2)}$, i.e. we *divide* the moduli and *subtract* the arguments.

Example 2: *Express (1 + i) ÷ (1 + 1.73i) in polar coordinates?*

We convert Cartesian to polar: For (1 + i) $\tan\theta = 1/1$ so $\theta = \pi/4$. For (1 + 1.73i) $\tan\Delta = 1.73/1$ so $\Delta = \pi/3$
For (1 +i) the value of $r^2 = 1^2 + 1^2$ so $r = \sqrt{2}$. For (1 + 1.73i) the value of $r^2 = 1^2 + (1.73)^2$ so r = 2.
So we can write $(1 + i) \div (1 + 1.73i) = (\sqrt{2})e^{i\pi/4} \div 2e^{i\pi/3} = 0.707e^{i\pi/4 - i\pi/3} = 0.707e^{-i\pi/12}$.

Powers and Roots

Again we *always* use the polar form. For a real number power it is straightforward: $z^n = r^n e^{in\phi}$.
i.e. we take the modulus to the n^{th} power and multiply the argument (or phase) by *n*.

Roots are trickier. We defined ϕ to lie in the region $0 < \phi < 2\pi$. But this will need to be extended if we want to get *all* the roots of a complex number. We define $z = re^{i(\phi + 2p\pi)}$ where *p* is an integer.

To find an n^{th} root, we need to take *n* distinct values of *p*: $p = 0$, $p = 1$, $p = 2$, ..., $p = n - 1$.
Then there are *n* distinct roots $z^{1/n} = r^{1/n} e^{i(\phi + 2p\pi)/n}$.

Example : I remember things better when I do them in easy small steps..so…if $z = 9\,e^{i\pi/3}$ what is $z^{½}$?

Step 1: write down z in polars with the 2πp bit added on to the argument. $z = 9\,e^{i(\pi/3 + 2\pi p)}$

Step 2: say how many values of p you'll need and write out the rooted expression ….. here n = 2 so I'll need 2 values of p; p = 0 and p = 1. $z^{½} = \sqrt{9}\,e^{i(\pi/3 + 2\pi p)/2}$

Step 3: Work it out for each value of p….$z^{½} = 3\,e^{i(\pi/3)/2} = 3\,e^{i(\pi/6)}$ for p = 0

$z^{½} = 3\,e^{i(\pi/3 + 2\pi)/2} = 3\,e^{i(\pi/6 + \pi)}$ for p = 1

There are your answers but remember that $e^{i\phi} = (\cos\phi + i\sin\phi)$ so $e^{i\pi} = (\cos\pi + i\sin\pi) = -1$.
It's therefore better to write $z^{½} = 3\,e^{i(\pi/6 + \pi)} = 3\,e^{i\pi/6}(e^{i\pi}) = -3\,e^{i\pi/6}$ for p =1, and $3\,e^{i(\pi/6)}$ for p = 0

Example 3: If $z = 27\,e^{i\pi/2}$ what is $z^{⅓}$?

Step 1: write down z in polars with the 2πp bit added on to the argument. $z = 27\,e^{i(\pi/2 + 2\pi p)}$

Step 2: say how many values of p you'll need and write out the rooted expression ….. here n = 3 so I'll need 3 values of p; p = 0, p = 1, and p =2. $z^{⅓} = \sqrt[3]{27}\,e^{i(\pi/2 + 2\pi p)/3}$

Step 3: Work it out for each value of p….$z^{½} = 3\,e^{i(\pi/2)/3} = 3\,e^{i(\pi/6)}$ for p = 0

$z^{½} = 3\,e^{i(\pi/2 + 2\pi)/3} = 3\,e^{i(\pi/6 + 2\pi/3)}$ for p = 1

$z^{½} = 3\,e^{i(\pi/2 + 4\pi)/3} = 3\,e^{i(\pi/6 + 4\pi/3)}$ for p = 2

So the answers are $3\,e^{i(\pi/6)}$ and $3\,e^{i(\pi/6 + 2\pi/3)}$ and $3\,e^{i(\pi/6 + 4\pi/3)}$

Exponentials and Trigonometric functions

Remember $e^{ikx} = \cos kx + i\sin kx$; $e^{-ikx} = \cos kx - i\sin kx$

Rearranging gives $\cos kx = \frac{1}{2}(e^{ikx} + e^{-ikx})$; $\sin kx = \frac{1}{2i}(e^{ikx} - e^{-ikx})$

This is a key observation…remember this.

Differentiation of a Complex Exponential

We know $\frac{d}{dx}e^{kx} = ke^{kx}$. Since *i* is just a constant, we similarly have $\boxed{\frac{d}{dx}e^{ikx} = ike^{ikx}}$

Note that is much nicer to differentiate exponentials than sines and cosines because we get exactly the same function as we started with, just multiplied by a constant.