Lecture 18: Spherical Polar Coordinates, continued

3. ∇^2 in Spherical Polars: Spherical Solutions

As given on the data sheet, $\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$

We'll look first at problems in which the solutions are known to be '*spherically symmetric*'. That is, the solutions depend on *r*, but have no angular dependence. They are functions of *r* but not of θ or ϕ.

For example if $F = F(r)$ then $\nabla^2 F(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}F(r)\right)$.

(a) The Laplace Equation $\nabla^2 V(r) = 0$.

Exercise *Find spherically symmetric solutions of Laplace's Equation* $\nabla^2 V(r) = 0$.

We have $\nabla^2 V(r) = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}V(r)\right) = 0$.

In this case we can actually find *V(r)* directly by rearranging and integrating, in steps.

Multiplying both sides by r^2 gives $\frac{d}{dr}\left(r^2\frac{d}{dr}V(r)\right) = 0$.

Integrating both sides gives $r^2\frac{d}{dr}V(r) = A$ where *A* is a constant. This rearranges to $\frac{d}{dr}V(r) = \frac{A}{r^2}$.

Integrate both sides again and we get the general solution: $V(r) = -\frac{A}{r} + B$.

<u>*Application*</u>

In electrostatics we want a potential which vanishes at ∞, so set $B = 0$, then $V(r) = -\frac{A}{r}$.

This is the standard Coulomb potential from a point charge at the origin: $V(r) = \frac{Q}{4\pi\varepsilon_0 r}$, with $A = \frac{-Q}{4\pi\varepsilon_0}$.

We have demonstrated not only that the Coulomb potential satisfies Laplace's equation but that this is the *only* spherically symmetric solution.

(b) The Wave Equation

In 3D the wave equation is $\nabla^2\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}$.

Let's only look for spherically symmetric solutions $\Psi(r,t)$, so the equation can be written

$$\nabla^2\Psi(r,t) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi(r,t)}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2\Psi(r,t)}{\partial t^2}$$

As in lecture 13 we look for solutions of the form $\Psi(r,t) = R(r)T(t)$, substitute this back in, and then separate the variables.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{T(t)dR(r)}{dr}\right) = \frac{1}{c^2}\frac{R(r)d^2T(t)}{dt^2} \quad \text{gives} \quad \frac{1}{R(r)r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right) = \frac{1}{c^2T(t)}\frac{d^2T(t)}{dt^2}$$

Each side of the equation must equal a constant, and we want oscillating solutions so we choose a negative constant. In order to help the maths let's set the constant as $-(\omega/c)^2$:

$$\frac{1}{R(r)r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)=\frac{1}{c^2T(t)}\frac{d^2T(t)}{dt^2}=-\left(\frac{\omega}{c}\right)^2$$

The equation for $T(t)$ is easy to solve. $\frac{d^2T(t)}{dt^2}=-c^2T(t)\left(\frac{\omega}{c}\right)^2=-\omega^2T(t)$ giving $T(t)\sim e^{\pm i\omega t}$.

Now we need to solve $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)=-\left(\frac{\omega}{c}\right)^2R(r)=-k^2R(r)$ (*) where $k^2=\frac{\omega^2}{c^2}$.

Equations like this occur frequently. There is a standard trick which is to define $R(r)=\frac{u(r)}{r}$, solve for *u(r)* and thus find *R(r)*.

Start by differentiating *R(r)* with respect to r using the product rule. $\frac{dR}{dr}=\frac{1}{r}\frac{du(r)}{dr}-u(r)\frac{1}{r^2}$

Multiply both sides by r^2 gives $r^2\frac{dR}{dr}=r\frac{du(r)}{dr}-u(r)$

Now differentiate, again using the product rule. $\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=r\frac{d^2u(r)}{dr^2}+\frac{du(r)}{dr}-\frac{du(r)}{dr}=r\frac{d^2u(r)}{dr^2}$

Therefore $\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR(r)}{dr}\right)=\frac{1}{r}\frac{d^2u(r)}{dr^2}$.

So equation (*) becomes: $\frac{1}{r}\frac{d^2u(r)}{dr^2}=-k^2\frac{u(r)}{r}$. The factors of r cancel, giving $\frac{d^2u(r)}{dr^2}=-k^2u(r)$.

Thus we have solutions of the form:

$$u(r)=Ae^{ikr}+Be^{-ikr} \qquad R(r)=\frac{Ae^{ikr}}{r}+\frac{Be^{-ikr}}{r} \qquad \Psi(r,t)=R(r)T(t)=\frac{\left(Ae^{ikr}+Be^{-ikr}\right)}{r}e^{i\omega t}.$$

For waves moving out from the origin $\Psi(r,t)=\frac{Ae^{ikr}e^{i\omega t}}{r}=\frac{Ae^{i(kr+\omega t)}}{r}$

For waves moving in towards the origin $\Psi(r,t)=\frac{Be^{-ikr}e^{i\omega t}}{r}=\frac{Be^{-i(kr-\omega t)}}{r}$

These are spherical waves moving in and out from the origin.
Note the factor of $1/r$. Intensity is related to amplitude squared. Our solution gives $|\Psi(r,t)|^2=A^2/r^2$.
This is the well known inverse square law.

Many other spherical equations and problems (e.g. heat flow in a sphere) can be solved in a similar way.