Lecture 13: The Wave Equation

Note: The rest of this course is *not* covered by Jordan & Smith. These notes are complete although you may find the Course Pack helpful (see Introduction section of these notes).

Introduction to PDEs

In many physical situations we encounter quantities which depend on two or more variables, for example the displacement of a string varies with space and time: $y(x, t)$. Handing such functions mathematically involves *partial differentiation* and *partial differential equations* (PDEs).

Revision of Partial Differentiation

Consider a function $y(x, t)$. Remember that to find $\frac{\partial y}{\partial x}$ (the partial derivative of y with respect to x), we differentiate with respect to x treating t as a constant.

Example Let $y = x^2 \sin t$. So $\frac{\partial y}{\partial x} = 2x\sin t$, $\frac{\partial y}{\partial t} = x^2 \cos t$, $\frac{\partial^2 y}{\partial x^2} = 2\sin t$ and $\frac{\partial^2 y}{\partial t^2} = -x^2 \sin t$.

Partial Differential Equations

Some of the most commonly occuring PDEs, and their areas of application, are listed below:

1	$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$	Wave equation	Elastic waves, sound waves, electromagnetic waves, etc.
2	$-\frac{\hbar^2}{2m}\nabla^2 u + Vu = i\hbar\frac{\partial u}{\partial t}$	Schrödinger's equation	Quantum mechanics
3	$\nabla^2 u = \frac{1}{h^2}\frac{\partial u}{\partial t}$	Diffusion equation	Heat flow, chemical diffusion, etc.
4	$\nabla^2 u = 0$	Laplace's equation	Electromagnetism, gravitation, hydrodynamics, heat flow.
5	$\nabla^2 u = -\frac{\rho}{\varepsilon_0}$	Poisson's equation	As (4) in regions containing mass, charge, sources of heat, etc.

Remember from last year that Gauss equation relates the surface integral of the electric field to the charge inside the surface. This can also be written as $\nabla.E = \frac{\rho}{\varepsilon_0}$ and since the electric field is related to the potential by $E = -\nabla V$ then we can write $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$. This is Poisson's equation, and in a charge free region of space this becomes Laplace's equation. This can be directly applied to fluid flow or gravitation by reassigning terms. The Schrödinger and diffusion equations will be covered in future lectures.

We will start by looking at equations 1-3 in one space dimension, then move on to 3D problems.
In many cases, solutions of PDEs can be found by separation of the variables. We will learn this method by considering waves on strings. In subsequent lectures we will use a similar procedure to solve many other PDEs.

The One-Dimensional Wave Equation

See Course Pack p.69-74, 95-101.

Waves on strings are governed by the equation $\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$, where $y(x, t)$ is the displacement of the string at position x and time t.

You will have met this equation briefly in Y1 and learned that $c^2 = T/\mu$ where μ is the mass per unit length of the string, T is the tension, and c is the wave velocity. In this course we will not be concerned with where the equation came from but only with finding its solutions, i.e. determining the motion of the string.

Consider the specific case of a string of length L attached at both ends to rigid supports. Then we additionally have the boundary conditions $y(0, t) = y(L, t) = 0$.

Note: A PDE can *never* be solved without knowing the boundary conditions!

Step 1: Separation of the Variables

Our boundary conditions are true at special values of x and for *all* values of time.
Also since y is a function of both x and t, then the solutions will be of the form $y(x,t) = X(x)T(t)$ where the big X and T are functions of x and t respectively.

Substituting this into the wave equation…

We have $\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[X(x)T(t)\right] = T(t)\frac{d^2 X(x)}{dx^2}$ and similarly $\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{c^2}X(x)\frac{d^2 T(t)}{dt^2}$.

So substitution gives $T(t)\frac{d^2 X(x)}{dx^2} = \frac{X(x)}{c^2}\frac{d^2 T(t)}{dt^2}$.

Rearrange the equation so all the terms in x are on one side and all the terms in t are on the other:

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = \frac{1}{c^2 T(t)}\frac{d^2 T(t)}{dt^2}$$

The *only* way this can be satisfied for all x, t is if *both* sides are equal to a constant:

$$\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = \frac{1}{c^2 T(t)}\frac{d^2 T(t)}{dt^2} = \text{constant}.$$

Suppose we call the constant N.

Then we have $\frac{1}{X(x)}\frac{d^2 X(x)}{dx^2} = N$ which rearranges to $\frac{d^2 X(x)}{dx^2} = N\,X(x)$. (i)

And $\frac{1}{c^2 T(t)}\frac{d^2 T(t)}{dt^2} = N$ which rearranges to $\frac{d^2 T(t)}{dt^2} = Nc^2\,T(t)$. (ii)

Now we have *ordinary differential equations* for $X(x)$ and $T(t)$ – which we can solve.

Consider the equation for $X(x)$. In lecture 3 we looked at two equations of this form:

1. $\frac{d^2}{dt^2}x(t) = -\omega_0^2 x(t)$ Linear harmonic oscillator
2. $\frac{d^2}{dt^2}x(t) = \alpha^2 x(t)$ Unstable equilibrium

Which case we have depends on whether our constant A is positive or negative. We need to make an appropriate choice for N by considering the physical situation, particularly the boundary conditions.

Step 2: Satisfying the Boundary Conditions

In our case the boundary conditions are $y(0, t) = y(L, t) = 0$. This means $X(0) = X(L) = 0$, i.e. X is equal to zero at *two* different points. This is crucial in determining the sign of A. Remember that (1) has oscillatory solutions (meaning that it will pass through zero displacement many times), whilst the solutions of (2) are exponential growth and decay that only tend to $x = 0$ once (see lecture 3 and compare a pendulum with a pencil falling from the vertical). From this we deduce that N *must be negative*. Let's write $N = -k^2$.

So (i) becomes $\frac{d^2 X(x)}{dx^2} = -k^2 X(x)$. From lecture 3, this has general solution $X(x) = A\cos kx + B\sin kx$.

Now we apply the boundary conditions:

$X(0) = 0$ gives $A = 0$.

We must take $B \neq 0$. So $X(L) = 0$ requires $\sin kL = 0$, i.e. $kL = n\pi$.
So k can only take certain values $k_n = n\pi/L$ where n is an integer (which we can chose to be positive)
So we have $X_n(x) = B_n \sin\frac{n\pi x}{L}$ for $n = 1, 2, 3, \ldots$.

The equations for $X(x)$ and $T(t)$ are equal to the *same* constant, so equation (ii) becomes

$$\frac{d^2T(t)}{dt^2} = N\,c^2\,T(t) = -k^2\,c^2\,T(t).$$

Looking at the diagram below $\lambda_n = \frac{2L}{n}$ and since $k_n = \frac{2\pi}{\lambda_n}$ then $k_n = \frac{n\pi}{L}$.

Since c is the wave velocity and $c = f\lambda$ then we can write $c = \frac{\omega\lambda}{2\pi} = \frac{\omega}{k}$ and so $\omega_n = k_n c$

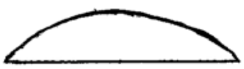
So we say $\frac{d^2T(t)}{dt^2} = -\omega_n^{\,2}\,T(t)$. This again has the form of the LHO equation.

Therefore it has solutions of the form $T_n(t) = (Ce^{i\omega_n t} + De^{-i\omega_n t})$ or $T_n(t) = (C\sin\omega_n t + D\cos\omega_n t)$ or $T_n(t) = C\cos(\omega_n t + \phi_n)$. (see lecture 3).

Hence we have special solutions:

$$y_n(x,t) = X_n(x)T_n(t) = B_n \sin k_n x \cos(\omega_n t + \phi_n) = B_n \sin\frac{n\pi x}{L}\cos\left(\frac{n\pi ct}{L} + \phi_n\right).$$

We see that each y_n represents harmonic motion with a different wavelength (different frequency). In the diagram below of course time is constant (as it's a photo not a movie!!):

$n = 1$		Fundamental	$k_1 = \frac{\pi}{L}$,	$\lambda_1 = \frac{2\pi}{k_1} = 2L$,	$\omega_1 = \frac{\pi c}{L}$
$n = 2$		1st harmonic	$k_2 = 2k_1 = \frac{2\pi}{L}$,	$\lambda_2 = \frac{\lambda_1}{2} = L$,	$\omega_2 = \frac{2\pi c}{L} = 2\omega_1$
$n = 3$		2nd harmonic	$k_3 = 3k_1 = \frac{3\pi}{L}$,	$\lambda_3 = \frac{\lambda_1}{3} = \frac{2}{3}L$,	$\omega_3 = 3\omega_1$, *etc.*

The Superposition principle
The wave equation (and all PDEs which we will consider) is a *linear* equation, meaning that the dependent variable and all its derivatives appear to the 1st power. For such equations there is a fundamental theorem called the **superposition principle**, which states that *if y_1 and y_2 are solutions of the equation then $y = c_1 y_1 + c_2 y_2$ is also a solution, for any constants c_1, c_2*. Put more simply this means that the net amplitude caused by two or more waves traversing the same space, is the sum of the amplitudes which would have been produced by the individual waves separately. Although this principle has been mainly used to describe constructive and destructive interference of waves, it was also used last year to describe net voltage within a circuit, energy transfer along a bar heated at both ends, and in the summation of the effects of charge distribution.

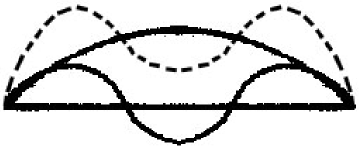
Step 3: Constructing the General Solution

Bearing in mind the superposition principle, the general solution of our equation is the sum of all special solutions:

$$y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\cos\left(\frac{n\pi ct}{L} + \phi_n\right).$$

This is the most general answer to the problem. For example, if a skipping rope was oscillated at both its fundamental frequency and its 2^{nd} harmonic, then the rope would look like the dashed line at some specific point in time and generally its displacement could be described by the equation :-

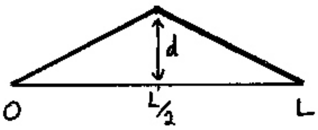
$$y(x,t) = B_1 \sin\frac{\pi x}{L}\cos\left(\frac{\pi ct}{L}+\phi_1\right) + B_3 \sin\frac{3\pi x}{L}\cos\left(\frac{3\pi ct}{L}+\phi_3\right)$$



NB. The Fourier series is a further example of the superposition principle.

Step 4: Solution of Complete Problem using Fourier Series

Suppose we have been given further information, namely we have been told that at time $t = 0$ the guitar string from lecture 8 example 5 is released from rest in the configuration shown below:



At $t = 0$ the string is at rest, i.e. $\left.\frac{\partial y}{\partial t}\right|_{t=0} = 0$, if we differentiate we find

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} -\frac{B_n n\pi c}{L}\sin\frac{n\pi x}{L}\sin\left(\frac{n\pi ct}{L}+\phi_n\right) = 0$$ so for this to be true $\phi_n = 0$ for all n.

So the general solution becomes $y(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$.

So at time $t = 0$, $y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$.

If we look back to lecture 8 example 5 we will see that the coefficients B_n are the coefficients of the Fourier series for the given initial configuration! We've already shown that the configuration drawn above at $t = 0$ can be expressed as a half-range sine series,

$$y(x,0) = \frac{8d}{\pi^2}\left[\sin\frac{\pi x}{L} - \frac{1}{9}\sin\frac{3\pi x}{L} + \frac{1}{25}\sin\frac{5\pi x}{L} - \frac{1}{49}\sin\frac{7\pi x}{L} + \ldots\right] \quad \text{for} \quad 0 \le x \le L.$$

Hence, by trusting the superposition principle and therefore treating each harmonic as a separate oscillating sinusoidal waveform, we deduce that at later times the configuration of the string will be:-

$$y(x,t) = \frac{8d}{\pi^2}\left[\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} - \frac{1}{9}\sin\frac{3\pi x}{L}\cos\frac{3\pi ct}{L} + \frac{1}{25}\sin\frac{5\pi x}{L}\cos\frac{5\pi ct}{L} - \frac{1}{49}\sin\frac{7\pi x}{L}\cos\frac{7\pi ct}{L} + \ldots\right].$$

SUMMARY of the procedure used:

1. We have an equation $\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2}$ with boundary conditions $y(0, t) = y(L, t) = 0$.
   We look for a solution of the form $y(x,t) = X(x)T(t)$.
   We find that the variables 'separate' $\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \frac{1}{c^2T(t)}\frac{d^2T(t)}{dt^2} = N$
2. We use the boundary conditions to deduce that N must be negative, i.e. $N = -k^2$.
   We use the boundary conditions further to find the allowed values of k and hence find $X(x)$.
   We find the corresponding solution of the equation for $T(t)$.
   Hence we can write down the special solutions.
3. By the principle of superposition, the general solution is a sum over all special solutions.
4. Given the initial configuration, or similar information, the Fourier series can be used to find the particular solution at all times.

In subsequent lectures we will use a similar procedure to solve other PDEs.

Try a few questions from 'Phil's Problems'.