Lecture 1: Revision of Algebra

Multiplying Brackets

All terms are included, e.g. $(x+y)^3 = (x+y)(x^2+2xy+y^2) = x(x^2+2xy+y^2)+y(x^2+2xy+y^2)$

Also $(x+a)(x-a) = x^2 - a^2$

You can check these by choosing a simple value for *x and y* in the above expression.

Binomial Series (See exam data sheet)

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + \binom{n}{k}x^n + \ldots, \text{ where } \binom{n}{k} = \frac{n!}{(n-k)!k!}.$$

- When n is a positive integer we have a finite series: i.e. the series terminates.
- When n is negative or non-integer, the series does *not* terminate.
- The series converges for all $|x| < 1$ since x to any power will be smaller than x.

The most useful job of both the binomial and Taylor series is to intelligently approximate an impossibly complicated expression to a few simple terms that pretty well equal the full solution. We do this when we say 'taking only the first two or three terms'. However we need to apply it correctly.

We can only approximate this, i.e. $(1+x)^n \approx 1+nx$ or $(1+x)^n \approx 1+nx+\frac{n(n-1)}{2}x^2$ **when x <<1.**

Consider $(a+b)^n$. This can be rewritten as $(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n = a^n(1+x)^n$ where $x = b/a$ The intelligent part is to choose *a, b* such that $|a| > |b|$ so $|x| << 1$.

How many terms you need to use depends on how small x is and on how accurately you need the answer. As a rule of thumb, in most physical applications it is fine to use these approximations for $x < 0.1$. Obviously if n is very large you need a correspondingly smaller value of x for rapid convergence.

Examples

1. *(a) Expand* $\frac{1}{1+x}$ *in powers of* x.

   We have $n = -1$, so $\frac{1}{1+x} = 1 + (-1)x + \frac{(-1)(-2)x^2}{2} + \frac{(-1)(-2)(-3)x^3}{3.2} + \ldots\ldots = 1 - x + x^2 - x^3 + ..$

   (b) If $x = 0.1$ *and we need accuracy to about* 1%, *how many terms do we need?*
   Since $x^3 \sim 10^{-3}$ we would guess we could stop at x^2. Let's check this:
   The exact answer is $(1+0.1)^{-1} = 0.9090909\ldots$. To 1% accuracy this is 0.91.
   The expansion above gave us:- first term = 1, second term = $-x$, third term = x^2, fourth term = $-x^3$
   So $(1+0.1)^{-1} = 1 - 0.1 + (0.1^2) - (0.1^3)$. Yes we can stop after two terms.

2. Rewrite $(\sin\theta + \cos\theta)^{15}$ for small θ using the binomial expansion for the first three terms.

   $[\cos\theta(\tan\theta+1)]^{15} = \cos^{15}\theta(\tan\theta+1)^{15}$ and $(\tan\theta+1)^{15} = 1 + 15\tan\theta + \frac{(15)(14)\tan^2\theta}{2!}$

   So $\cos^{15}\theta(\tan\theta+1)^{15} = \cos^{15}\theta + 15\cos^{15}\theta\tan\theta + 105\cos^{15}\theta\tan^2\theta$

NB. The binomial expansion works for any value of x and n. It's just more useful if x <<1

Taylor Series (also on the exam data sheet)

$$f(x+a) = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \ldots + \frac{f^n(a)}{n!}x^n + \ldots \qquad \text{where of course } f^n(a) = \left.\frac{d^n f}{dx^n}\right|_{x=a}$$

Unlike the binomial expansion, the Taylor expansion can also be used for *any* function that has a derivative. Note that if $a = 0$ then the Taylor expansion is known as a Maclaurin expansion.

The range of x for which there is convergence depends on the function f. But in practice the series is only *useful* if the first couple of terms give an adequate approximation, which means we need x<<1.

Examples

1. Find the Maclaurin expansion of $e^{\alpha x}$?

$$f(x) = \exp(\alpha x) \qquad f'(x) = \alpha\exp(\alpha x) \qquad f''(x) = \alpha^2\exp(\alpha x)$$

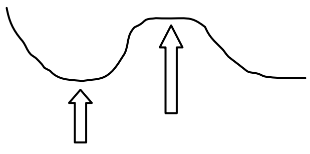
$$\exp(\alpha x) = \exp(0) + \alpha\exp(0)x + \frac{\alpha^2\exp(0)x^2}{2!} + \frac{\alpha^3\exp(0)x^3}{3!} + \ldots = 1 + \alpha x + \frac{\alpha^2 x^2}{2} + \ldots$$

2. Find the first two terms of the expansion of $\tan(\frac{\pi}{4} + x)$.

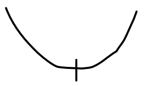
$$\tan(\frac{\pi}{4} + x) = \tan\frac{\pi}{4} + x\frac{d\tan x}{dx} + \ldots.. = \tan\frac{\pi}{4} + \sec^2 x\Big|_{x=\pi/4} + \ldots = \tan\frac{\pi}{4} + \frac{1}{\cos^2 \pi/4} + \ldots$$

Now $\tan\frac{\pi}{4} = 1$, $\qquad \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ $\qquad$ so $\qquad .\tan(\frac{\pi}{4} + x) = 1 + 2x$

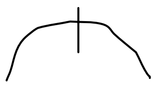
Why do we use these expansions so often in physics? Because we like to solve *easy* problems.
E.g. Stability
Suppose a particle of mass m lies on a potential surface $V(x)$ at $x = x_0$. There is no resultant force on it at this point – i.e. the particle is in equilibrium. This would be true if x_0 were at either of the positions marked with arrows in the figure.



The condition that there is no force at x_0 is that $\left.\frac{dV(x)}{dx}\right|_{x=x_0} = 0$. But we want to know whether the equilibrium is stable! We find this by asking if the potential increases or decreases as we move away from x_0. This is equivalent to determining whether $\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_0}$ is positive or negative.



$\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_0} > 0$ Stable



$\left.\frac{d^2V(x)}{dx^2}\right|_{x=x_0} < 0$ Unstable

Finding and differentiating a complete expression for $V(x)$ might be very hard, but an approximate expression valid near x_0 is all we need to answer our question.